THE CARAVAN INTERMEDIATE TRIGONOMETRY

text

by

GHULAM ABBAS KHAN, M.A., P.E.S.

and

RANA RASHEED AHMAD KHAN, M.A.

FOURTH EDITION

THE CARAVAN BOOK HOUSE
AIBAK ROAD - ANARKALI - LAHORE

THE CARAVAN INTERMEDIATE TRIGONOMETRY

by

GHULAM ABBAS KHAN, M.A., P.E.S.

Principal, Dyal Singh College, Lahore

FORMERLY PRINCIPAL, GOVERNMENT COLLEGE, DEHRA GHAZI KHAN
& MEMBER BOARD OF STUDIES IN MATHEMATICS &
ASTRONOMY, UNIVERSITY OF THE PANJAB

and

RANA RASHEED AHMAD KHAN, M.A.

LECTURER IN MATHEMATICS, GOVT. COLLEGE, RAWALPINDI

FOURTH
EDITION

THE CARAVAN BOOK HOUSE

AIBAK ROAD - ANARKALI - LAHORE

Price: Rs. 5/- only

1958

PRINTED AND PUBLISHED BY CH. ABDUL HAMID, M.A., AT THE CARAVAN PRESS, AIBAK ROAD, ANARKALI, LAHORE

PREFACE TO THE FOURTH EDITION

The book has been thoroughly revised and many useful changes have been made. The number of solved examples is largely increased and hints to various difficult questions have been added.

Answers to questions have been given at the end of each chapter.

It is hoped that the book in its present form shall be found helpful by the students in every respect.

Authors.

CONTENTS

LIST OF FORMULÆ AND IMPORTANT RESULTS

Chap:er I

1 right angle $= 90^\circ$, $1^\circ = 60'$, $1' = 60''$

1 right angle $= 104^g$, $1^g = 100'$, $1' = 100''$

$\pi = \frac{22}{7} = \frac{355}{113} = 3{\cdot}1416$ nearly

$1^c = 57^\circ\ 17'\ 44{\cdot}8''$

2 right angles $= \pi$ radians $= 180$ degrees $= 200$ grades

Circular measure of an angle $= \dfrac{\text{arc}}{\text{radius}} \left[\theta = \dfrac{l}{r}\right]$

Chapter II

$\sin^2\theta + \cos^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \text{cosec}^2\theta.$

$\sin\theta$, $\cos\theta$ are never greater then 1 numerically

$\sec\theta$, $\text{cosec}\,\theta$ are never less than 1 numerically

$\tan\theta$, $\cot\theta$ can have any value.

Chapter III

angle	0°	30°	45°	60°	90°
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Chapter IV

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$

$\sin(90-\theta) = \cos\theta$	$\sin(90+\theta) = \cos\theta$
$\cos(90-\theta) = \sin\theta$	$\cos(90+\theta) = -\sin\theta$
$\tan(90-\theta) = \cot\theta$	$\tan(90+\theta) = -\cot\theta$
$\sin(180-\theta) = \sin\theta$	$\sin(180+\theta) = -\sin\theta$
$\cos(180-\theta) = -\cos\theta$	$\cos(180+\theta) = -\cos\theta$
$\tan(180-\theta) = -\tan\theta$	$\tan(180+\theta) = +\tan\theta$

$\sin(360+\theta) = \sin\theta$; $\cos(360+\theta) = \cos\theta$,
$\tan(360+\theta) = \tan\theta$

Chapter VI

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\cot(A+B) = \frac{\cot A\cot B - 1}{\cot B + \cot A}$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$
$$\cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A}.$$
$$\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$$
$$\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$$
$$\tan(A+45^\circ) = \frac{1+\tan A}{1-\tan A}.$$
$$\tan(A-45^\circ) = \frac{\tan A - 1}{\tan A + 1}.$$
$$\tan(A+B+C) = \frac{\Sigma\tan A - \tan A\tan B\tan C}{1 - \Sigma\tan A\tan B}$$
$$\tan 75^\circ = 2+\sqrt{3}\ ;\ \tan 15^\circ = 2-\sqrt{3}.$$

Chapter VII

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$2\sin A\sin B=\cos(A-B)-\cos(A+B).$$

$$\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}.$$

Chapter VIII

$$\sin 2A=2\sin A\cos A=\frac{2\tan A}{1+\tan^2 A}.$$

$$\cos 2A=\cos^2 A-\sin^2 A=1-2\sin^2 A=2\cos^2 A-1$$
$$=\frac{1-\tan^2 A}{1+\tan^2 A}.$$

$$\sin 3A=3\sin A-4\sin^3 A$$

$$\cos 3A=4\cos^3 A-3\cos A$$

$$\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$$

$$\sin 18^\circ=\frac{\sqrt{5}-1}{4}$$

$$\cos 36^\circ=\frac{\sqrt{5}+1}{4}$$

$$\sin A=\pm\sqrt{\frac{1-\cos 2A}{2}}$$

$$\cos A=\pm\sqrt{\frac{1+\cos 2A}{2}}$$

$$\tan A=\pm\sqrt{\frac{1-\cos 2A}{1+\cos 2A}}$$

$$\sin A+\cos A=\pm\sqrt{1+\sin 2A}$$

$$\sin A-\cos A=\pm\sqrt{1-\sin 2A}.$$

$\sin\frac{A}{2}+\cos\frac{A}{2}$ has the same sign as $\sin\left(\frac{A}{2}+45^\circ\right)$

$\sin\frac{A}{2}-\cos\frac{A}{2}$ has the same sign as $\sin\left(\frac{A}{2}-45^\circ\right)$

Chapter IX

If $\sin\theta=0$, $\theta=n\pi$

$\cos\theta=0$, $\theta=(2n+1)\frac{\pi}{2}$

$\sin\theta=\sin\alpha$, $\theta=n\pi+(-1)^n\alpha$

$\cos\theta=\cos\alpha$, $\theta=2n\pi\pm\alpha$

$\tan\theta=\tan\alpha$, $\theta=n\pi+\alpha$.

Chapter X

If any $\triangle ABC$

$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ (sine Formula).

$\cos A=\frac{b^2+c^2-a^2}{2bc}$ (cosine Formula).

$\cos B=\frac{c^2+a^2-b^2}{2ca}$

$\cos C=\frac{a^2+b^2-c^2}{2ab}$.

$a=b\cos C+c\cos B$ (Projection formula).

$b=c\cos A+a\cos C$

$c=a\cos B+b\cos A$.

$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$ (Napier's analogy).

$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$, etc. (Semi-sum formula).

$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$, etc. (,, ,,)

$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ etc. (,, ,,)

$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$.

Chapter XI

$\log_a 1 = 0,\ \log_a a = 1,$

$\log_a m.n = \log_a m + \log_a n$

$\log_a \dfrac{m}{n} = \log_a m - \log_a n$

$\log_a m^n = n \log_a m$

$\log_a m = \log_b m \times \log_a b$ (change of base)

$\log_b a \times \log_a b = 1.$

Chapter XIV

Area of a triangle

$$\triangle = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}ca \sin B = \tfrac{1}{2}ab \sin C$$
$$= \sqrt{s(s-a)(s-b)(s-c)}$$

Formulae for Circum-radius

$$R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}$$
$$= \frac{abc}{4\triangle}$$

Formulae for In-radius

$$r = \frac{\triangle}{s}$$
$$= (s-a) \tan \frac{A}{2} = \ldots\ldots\ldots$$

Formulae for Ex-radius

$$r_1 = \frac{\triangle}{s-a}$$
$$= s \tan \frac{A}{2} = \ldots\ldots\ldots$$

Chapter XV

$$R = \frac{a}{2 \sin \frac{\pi}{n}}$$

$$r = \frac{a}{2 \tan \frac{\pi}{n}}$$

Area of a regular polygon of n sides

$$=\frac{nR^2}{2}\sin\frac{2\pi}{n}$$

$$=nr^2\tan\frac{\pi}{n}$$

$$=\frac{na^2}{4}\cot\frac{\pi}{n}.$$

$\sin\theta<\theta<\tan\theta$ where θ is acute.

$$\underset{\theta\to 0}{\mathrm{Lt}}\ \frac{\sin\theta}{\theta}=1$$

Area of a circle $=\pi R^2$

Circumference of a circle $=2\pi R$.

CHAPTER I

INTRODUCTION

1. The word 'Trigonometry' is derived from two Greek words—*Trigonon* (meaning a 'triangle') and '*metron*' (meaning 'I measure') and hence it means the measurement of angles and sides of triangles, and investigation of relations that exist between them. Trigonometry has two branches.

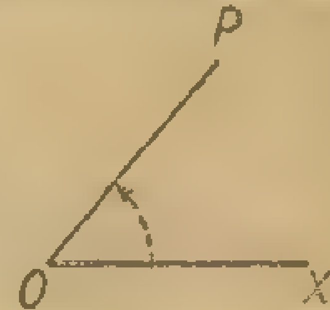
(*i*) Plane Trigonometry and (*ii*) Spherical Trigonometry.

Plane Trigonometry deals with triangles drawn in a plane, while **Spherical Trigonometry** has for its aim the solution of triangles drawn on a spherical surface.

In modern times, 'Plane Trigonometry' has a wider meaning. It is no longer restricted to the solution of triangles but comprises all alegbraical investigations with respect to plane angles whether those angles are parts of a triangle or not.

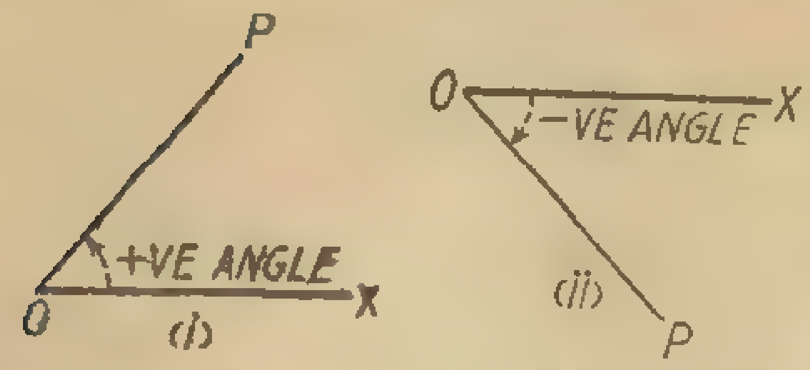
1·2. Angles :—An angle has been defined by Euclid as an inclination between two intersecting straight lines. This definition is too narrow as it does not include angles greater than two right angles. We shall have to extend our definition.

Def. An angle is the amount of revolutions when a line revolving about one of its extremities, passes from one position to another.



Thus if a line revolves about one extremity O and takes up the position OP from its initial position OX, it traces out angle XOP. OX is called the initial line.

2·1. Positive and Negative Angles :—If the revolving line OP revolves in the counter-clockwise direction (the direction opposite to that in which the hands of a clock revolve), it des-



cribes a positive angle. [Note fig. (*i*)]. If the revolving line OP revolves in the clockwise direction (the direction of revolution of the hands of a clock) it describes a negative angle [Note fig. (*ii*)].

1·21. Magnitude of positive angles :—Let X′OX and Y′OY be two ⊥ lines meeting at O When the revolving line

occupies the position OP between OX and OY, it is said to have described an acute angle or an angle less than a right angle; when it coincides with OY the angle described is equal to one right angle; when it is in the position OQ between OY and OX′, the angle is obtuse or an angle less than two right angles and greater than a right angle; when it coincides with OX′ the angle described is equal to two right angles; when it is in the position OR between OX′ and OY′, the angle is greater than two right angles and less than three right angles; when it coincides with OY′, the angle is equal to three right angles; when it is in the position OS between OY′ and OX, the angle is greater

than three right angles and less than four right angles; when it coincides with OX again after a complete revolution, the angle is equal to four right angles.

If after coinciding with OX, the line still revolves, the angle described will be greater than four right angles and it will describe as many 4 right angles as there are complete revolutions.

1·22. Magnitude of negative angles.

(*a*) When the revolving line OP revolves in the clockwise direction and takes up the position OP between OX and OY′, the angle described lies between 0 and $-90°$ and when it coincides with OY′ the angle described is $-90°$.

(*b*) When the revolving line takes up the position OQ between OY′ and OX′ the ∠ described lies between $-90°$ and $-180°$ and when it coincides with OX′ the ∠ described is -180.

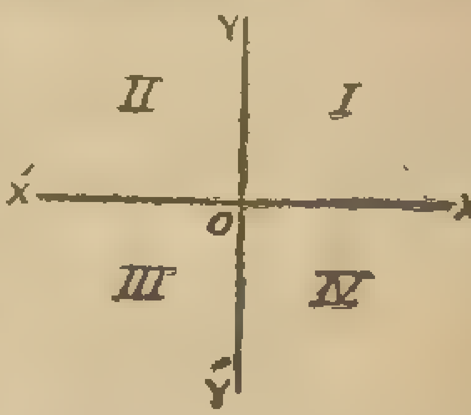
(*c*) When the revolving line takes up the position OR, the ∠ described lies between $-180°$ and $-270°$ and when it coincides with OY the ∠ described is $-270°$.

(*d*) When the revolving line takes up the position of OS between OY and OX, the ∠ described lies between $-270°$ and $-360°$ and when it coincides with OX, the ∠ described is $-360°$.

If, however, after coinciding with OX, the line revolves still further the angle described is beyond $-360°$.

From the above discussion it is clear that there is no limit to the magnitude of an angle, whether positive or negative, as there is no restriction to the number of revolutions of the revolving line OP. An angle may thus assume infinite positive or negative values.

1·3. Quadrants :—Let XOX′ and YOY′ be two ⊥ lines



meeting at O. They divide the plane into four parts. The part XOY is known as the first quadrant. YOX′ is known as the second quadrant, X′OY′ is known as the third quadrant and YOX as the fourth quadrant.

In the first quadrant the angle varies from 0° to one rt. ∠.

In the 2nd quadrant the angle varies from one rt. ∠ to 2 rt. ∠s.

In the 3rd quadrant the angle varies from 2rt. ∠s to 3rt. ∠s.

In the 4th quadrant the angle varies from 3rt. ∠s to 4rt. ∠s.

Example (*i*) :—*In which quadrant is the revolving line when it has traced out an angle of* (*i*) *1020°* (*ii*) *−560°*.

Sol. (*i*) $1020° = 2 \times 360° + 300°$.

$\therefore$ the revolving line has made two complete revolutions and further traced out an $\angle$ of 300°, which lies between 270° and 360° and hence the revolving line is in the fourth quadrant.

Sol. (*ii*) $-560 = -360° + (-200°)$.

$\therefore$ the revolving line has revolved in the clockwise direction and after having made one complete revolution makes further an angle of −200°, which lies between −180° and −270° and hence the revolving line lies in the 2nd quadrant.

Example (*ii*) :—*Prove that the position of the revolving line when it has turned through 930° is the same when it has turned through an angle −510°.*

Sol. $930 = 2 \times 360 + 210°$.

$\therefore$ the revolving line lies in the 3rd quadrant and has an $\angle$ of 30° beyond the line OX′.

Now $-510° = -360° + (-150°)$.

$\therefore$ the revolving line has made a complete revolution in the clockwise direction and has made an $\angle$ of 150° in the same direction and hence it also lies in the 3rd quadrant and it makes an $\angle$ of 60° beyond OY′.

$\angle X'OY' = 90°$.

$\therefore$ both positions are the same.

EXERCISES

1. Find in which quadrant the revolving line lies if it has turned through an angle,

(*i*) 1125°. (*ii*) −760.

2. Prove that the position of the revolving line is the same when it has turned through an angle of 675° or −405°.

1·4. Measurement of Angles :—Now in order to find the magnitude of angles, we must fix some unit with which to measure them. In geometry it is customary to measure an angle in terms of a rt. angle, but in trigonometry, there are three systems of measuring angles :—

I. Sexagesimal System or English System.

II. Centesimal System or French System.

III. Circular System :

1·41. I. Sexagesimal System :—What is the Sexagesimal system ? *(P. U. 1944)*

In this system a right angle is subdivided as follows :—
1 right angle = 90 degrees (written as $90°$)
1 degree $(1°)$ = 60 minutes (,, $60'$)
1 minute $(1')$ = 60 seconds (,, $60''$)

The unit in this case is one degree, which is divided into sixty equal parts, called minutes ; and each of these sixty parts is again subdivided into sixty equal parts, called seconds. This explains the name of the system, because in Latin 'sexaginta' means sixty and 'sexagesimus' means 60th and hence sexagesimal is sixtieth, of 60, proceeding by sixties.

3·2. II. Centesimal System :—What is the Centesimal System ? *(P. U. 1944)*

In this system a right angle is subdivided as follows :
1 right angle = 100 grades (written as 100^g)
1 grade (1^g) = 100 minutes (,, $100^`$)
1 minute $(1^`)$ = 100 seconds (,, $100^{``}$)

The unit used in this system is a grade, which is divided into 100 equal parts, called minutes ; and each of these 100 parts is again subdivided into 100 equal parts, called seconds. This explains the name of this system as 'centum' meaning hundred ; 'centesimus' means hundredth and hence 'centesimal' means reckoning by hundredths.

Note the difference in the symbols used to denote minutes and seconds in the two systems.

Out of these two systems, the Sexagesimal System is most commonly used in practice, although the Centesimal System is easily convertible into decimal fractions, for a division by hundred simply changes the place of the decimal to two places to the left and a multiplication by hundred changes the place of the decimal to two places to the right. The Centesimal System is now almost obsolete and is not even in vogue in France, where it was developed during the French Revolution,

because all mathematical and astronomical tables had already been calculated before its adoption.

1·5. To convert Sexagesimal into Centesimal Measure and *vice versa*.

From the above discussion we find that a right angle is a connecting link between the two systems and it is possible to convert the measure of an angle from one system to the other.

We shall, therefore, use 1 rt. $\angle = 90^\circ = 100^g$.

If the angle contains minutes and seconds,

(*a*) In the Sexagesimal System, we first reduce it to degrees and then use the above relation ;

(*b*) In the Centesimal System, we first reduce it to grades and then use the above relation.

In numerical problems it is generally found more convenient to reduce an angle as a decimal fraction of a right angle and then reduce it to the required system.

Example 1. *Reduce 71° 47′ 51″ to Centesimal Measure*

$51'' = \frac{51'}{60} = {\cdot}85'$ (dividing by 60 to make minutes of 51″)

$$47'\ 51'' = 47{\cdot}85' = \frac{47{\cdot}85^\circ}{60} = {\cdot}7975^\circ$$

$$\therefore \quad 71^\circ\ 47'\ 51'' = 71{\cdot}7975^\circ = \frac{71{\cdot}7975}{90} \text{ right angles}$$

$$= {\cdot}79775 \quad \text{,,} \quad \text{,,}$$

$\therefore \quad 71^\circ\ 47'\ 51'' = {\cdot}79775$ rt. angles.

We notice here that the Sexagesimal measure has been reduced to a decimal fractionof a right angle.

Now ·79775 right $\angle$s $= {\cdot}79775 \times 100$ grades

$= 79{\cdot}775$ grades $\quad [\because 1 \text{ rt. } \angle = 100^\circ]$.

We leave alone 79 and convert the decimal part ·775 into minutes by multiplying it by 100.

$\therefore \quad {\cdot}775$ grades

$= \frac{100}{77{\cdot}5}$ minutes.

We leave alone 77 and convert ·5 minutes into seconds by multiplying it by 100.

$\therefore$ ·5 minutes

$= \frac{100}{50}$ seconds.

$\therefore\quad 71^\circ\ 47'\ 51'' = 79^g\ 77^{\backprime}\ 50^{\backprime\backprime}$.

In practice the above process after the fraction of a rt. $\angle$ could be worked as follows.

$71^\circ\ 47'\ 51'' =$ ·79775 right angles
100
79·775 grades
100
77·5 minutes
100
50 seconds

$\therefore\quad 71^\circ\ 47'\ 51'' = 79^g\ 77^{\backprime}\ 50^{\backprime\backprime}$.

Example 2. *Reduce $48^g\ 77^{\backprime}\ 15^{\backprime\backprime}$ to Sexagesimal System.*

$$15^{\backprime\backprime} = \frac{15^{\backprime}}{100} = {·}15^{\backprime}$$

$$77^{\backprime}\ 15^{\backprime\backprime} = 77{·}15^{\backprime} = \frac{77{·}15^g}{100} = {·}7715^g$$

$$\therefore\quad 48^g\ 77^{\backprime}.15^{\backprime\backprime} = 48{·}7715^g = \frac{48{·}7715}{100} \text{ rights } \angle\text{s}$$

= ·487715 right $\angle$s
90
43·89435 degrees
60
53·66100 minutes
60
39·660 seconds

$\therefore\quad 48^g\ 77^{\backprime}\ 15^{\backprime\backprime} = 43^\circ\ 53'\ 39{·}66''$.

EXERCISE (a)

1. Express $23^\circ\ 28'\ 45''$ in grades, minutes and seconds (Centesimal).

2. Express $26^g\ 8^{\backprime}\ 79{·}5^{\backprime\backprime}$ in degrees, minutes and seconds (Sexagesimal).

3. Find the number of quadrants in which the ∠s 1546^g lie.

[Hint A complete revolution makes $\left(360\times\frac{100}{90}\right)^g$ or 400^g.

∴ each quadrant has 100^g in it.

Now $1546^g = 3\times 400^g + 346^g$.

∴ it is three complete revolutions $+346^g$, which lie in the 4th quardrant, etc.

4. The number of degrees in one acute angle of a right ∠d triangle is equal to the number of grades in the other, find both the angles in degrees.

[Hint: Let one $\angle = x^\circ$; then the other $\angle = (90-x)^\circ$ $= (90-x)\frac{100^g}{90} = (90-x)\frac{10^g}{9}$. Their numbers are equal

∴ $x = (90-x)\frac{10}{9}$ etc.].

5. If the number of degrees in an angle of a △ is less than 10, then the number of grades in that angle show that the △ is a rt. ∠d one.

6. Find the number of sides in a regular polygon, each of whose angles is 150^g.

[Hint: $150^g = 150\times\frac{9^\circ}{10} = 135^\circ$, let the side be n in uumber

∴ the sum of the interior ∠s $=(135\times n^\circ)$ but all interior ∠s of a polygon $=(2n-4)$ rt. ∠s $=(2n-4)\times 90^\circ$.

∴ $135\times n = (2n-4)\ 90$ etc.

or exterior $\angle = 180 - 135^\circ = 45^\circ$: sum of exterior $\angle = 360^\circ$

∴ Number of sides $=\frac{360}{45} = 8$].

7. The sum of two angles is 80 grades and their difference is 18 degrees. Find the angles in degrees. (*P. U. 1944*)

Hint: Let the ∠s be x°, y°. Their sum is given in grades; we convert them into degrees, because we have supposed the ∠s in degrees.

$$\text{Sum} = 80^g = 80\times\frac{9^\circ}{10} = 72^\circ.$$

∴ $x^\circ + y^\circ = 72^\circ$ or $x+y=72$; also $x-y=18$.

Find x and y by solving the two equations.

Note it is always convenient to suppose the $\angle$s in degrees, as the students are more familiar with degrees.]

8. The difference of two angles is $11\frac{1}{9}^{g}$ and their sum is 60°, find the angles.

9. If M_1 and M_2 denote respectively the number of English or French minutes in any angle, prove that

$$\frac{M_1}{27}=\frac{M_2}{50}.$$

[**Hint** : Let $\angle$ be x° $\therefore =(x\times 60)'$ $\therefore$ $M_1=x\times 60$ ... (i)

Also $x^\circ=\left(x\times\frac{10}{9}\right)^g=\left(x\times\frac{10}{9}\times 100\right)'$

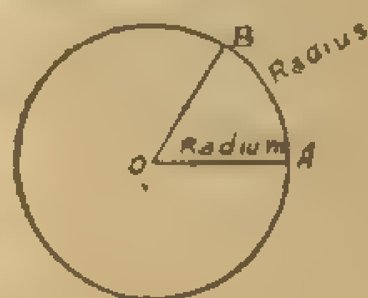
$\therefore$ $M_2=x\times\frac{10}{9}\times 100$ ..(ii). Dividing (i) by (ii)

$$\frac{M_1}{M_2}=\frac{x\times 60}{x\times\frac{10}{9}\times 100}=\frac{27}{50}\therefore\frac{M_1}{27}=\frac{M_2}{50}.\ ]$$

10. If S_1 and S_2 denote respectively the number of English and French seconds in any angle, prove that $\frac{S_1}{81}=\frac{S_2}{250}$. (*P.U.*)

1·6. III Circular System. What is the Circular System ? (*P.U. 1944*)

The unit adopted in this system is called a **Radian.** It is defined to be *the angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle.* (In the figure, arc AB=radius OA. $\angle$AOB is the radian.) And the circular measure of an angle is the number of radians it contains.



(Note that the arc AB and not the chord AB is equal to the radius).

It now remains to be shown that radian is a constant angle, whatever be the radius of the circle for a unit employed, is always a constant quantity.

1·61. To prove this we have to prove **that the circumference of a circle bears a constant ratio to its diameter.**

Draw any two circles with centres O and O′ and radii r, r'.

Let c, C′ be their circumferences.

Inscribe a regular polygon of n sides in each circle. Let AB and A′B′ be the sides of the two polygons. Join OA, OB, O′A′ and O′B′.

In the Δs OAB, O′A′B′

$\angle AOB = \angle A'O'B'$ $\quad$ ($\because$ each $= \dfrac{1}{n} \times 4$ right angles)

$\dfrac{OA}{O'A'} = \dfrac{OB}{O'B'}$ $\quad$ ($\because$ OA=OB, OA′=OB′)

$\therefore$ Δs are similar ; $\quad \therefore \dfrac{AB}{A'B'} = \dfrac{OA}{O'A'}$

Now $\dfrac{\text{Perimeter of the first polygon}}{\text{Perimeter of the second polygon}} = \dfrac{n.AB}{n.A'B'} = \dfrac{AB}{A'B'}$

$$= \frac{OB}{O'A'} \quad \dots (1)$$

This is true whatever be the number of sides of the two polygons.

If the number of sides of the polygons be increased indefinitely, the perimeter of the polygons will practically become the circumference of the two circles.

$\therefore$ from (1) $\dfrac{\text{circumference of first circle}}{\text{circumference of second circle}} = \dfrac{OA}{O'A'} = \dfrac{r}{r'}$

$i.e.,\ \dfrac{c}{c'} = \dfrac{r}{r'}$ or $\dfrac{c}{r} = \dfrac{c'}{r'}$ $\quad \therefore \dfrac{c}{2r} = \dfrac{c'}{2r'}$

$\therefore$ The ratio $\dfrac{\text{circumference of any circle}}{\text{diameter}}$ is the same for all circles.

This constant ratio is universally denoted by the Greek letter π (pronounced as 'Pi').

Cor. Circumference of the circle of radius $R = 2\pi R$

1·62. The value of π has been calculated to 700 places of decimals. Its value correct to 8 places of decimals is 3·14159265.

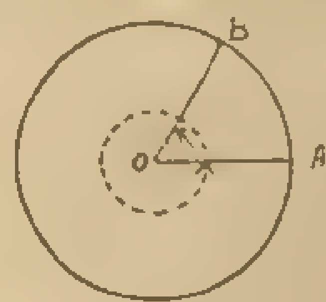
For all practical purposes, the following approximate values of π are generally sufficient : $\dfrac{22}{7}$, $\dfrac{355}{113}$, 3·1416.

Note The value : $\dfrac{22}{7}$ is correct up to 2 places of decimal and $\dfrac{355}{133}$ is correct up to 6 places of decimal.

1·7. The Radian is a constant angle.

(*P.U. 1934, 1937, 1941, 46, 47. Supp., 48*)

Take AB, an arc of a circle whose length is equal to the radius of the circle.



Then $\dfrac{\angle AOB}{4 \text{ right angles}}$

$$= \frac{\text{arc AB}}{\text{circumference of the circle}}$$

[$\because$ the arcs of a circle are proportional to the angles they subtend at the centre]

or $$\frac{\angle AOB}{4 \text{ right angles}} = \frac{r}{2\pi r} = \frac{1}{2\pi}$$

$$\therefore \quad \angle AOB = \frac{4 \text{ right angles}}{2\pi} = \frac{2 \text{ right angles}}{\pi}$$

$$= \text{constant}$$

Thus the $\angle AOB$ or radian is an angle whose magnitude is independent of the radius, and is therefore a constant quantity.

Note. $1 \text{ radian} = \dfrac{2 \text{ right angles}}{\pi}$

i.e., π radians $= 2$ right angles $= 180^\circ = 200^g$,

The letter sin C is generally used to denote a radian.

1·71. To find the value of a Radian.

(*P.U. 1946, 48*)

$$1 \text{ Radian} = \frac{2 \text{ right angles}}{\pi}$$

$$= \frac{180^\circ}{\pi}$$

$$= 180 \times \frac{113}{355} \text{ degrees}$$

Taking $\pi = \frac{355}{113}$ to secure greater accuracy]

$$= \frac{4068^\circ}{71}$$

$$= 57017'44{\cdot}8''$$

Taking the result in seconds correct to one place of decimal.]

$$\begin{array}{r|l}
 & 57^\circ \\
71) & 4068 \\
 & 355 \\
 & 518 \\
 & 497 \\
 & 21 \\
 & 60 \\
71) & 1260 \quad (17 \text{ minutes} \\
 & 71 \\
 & 550 \\
 & 497 \\
 & 53 \\
 & 60 \\
71) & 3180 \quad (44{\cdot}78 \\
 & 284 \\
 & 340 \\
 & 284 \\
 & 560 \\
 & 497 \\
 & 630 \\
 & 568
\end{array}$$

EXAMPLES

1. Show that a radian contains $206265''$ approximately.

(*P.U. 1935, 48*)

Hint $1^\circ = \frac{180^\circ}{\pi} = \frac{4068^\circ}{71} = \frac{4068}{71} \times 60 \times 60''$ and simplify]

2. Show that $1^\circ = {\cdot}017453$ radian nearly :

Sol. $180 = \pi$ radians $1^\circ = \frac{\pi}{180}$ radians $= \frac{3{\cdot}14159}{180}$

$= {\cdot}017453$ radians nearly]

1·72. To convert Sexagesimal and Centesimal measure into circular measure and *vice versa* :

We use the relation π radius $= 180^\circ = 200^g$

ILLUSTRATIVE EXAMPLES

1. *Express 30° and 750° in radians.*

Sol. Since $180° = \pi$ radians.

$\therefore \quad 30° = \frac{\pi}{180} \times 30 = \frac{\pi}{6}$ radians

Similarly $750° = \frac{\pi}{180} \times 750$ radians

$= \frac{25\pi^c}{6}.$

2. *Express 50^g and 150^g in radians and degrees.*

Since $200^g = \pi$ radians

$\therefore \quad 50^g = \frac{\pi}{200} \times 50$ radians

$= \frac{\pi}{4}$ radians

$= \frac{\pi}{4} \times \frac{180}{\pi} \quad (\because \ \pi \text{ radians} = 180°)$

$= 45°$

Similarly $150^g = \frac{\pi}{200} \times 150$ radians

$= \frac{3\pi}{4}$ radians

$= \frac{3\pi}{4} \times \frac{180}{\pi}$ degrees

$= 135°.$

3. *Express (a) 39° 22′ 30″ (b) 40^g 37\` 15\`\` in radians.*

(a) $39° \ 22' \ 30'' = 39° 22\frac{1}{2}' = 39\frac{3}{8}° = \frac{315°}{8}$

Now $\quad 180° = \pi$ radians

$\therefore \quad \frac{315°}{8} = \frac{315}{8} \times \frac{\pi}{180}$ radians

$= \frac{7\pi^c}{32}.$

(b) 40^g 37\` 15\`\` $= 40^g$ 37·15\` $= 40{\cdot}3715^g$

$\therefore \quad$ 200 grades $= \pi$ radians

$$\therefore \quad 40{\cdot}3725^g = 40{\cdot}3715 \times \frac{\pi}{200} \text{ radians}$$
$$= {\cdot}2018575\pi \text{ radians.}$$

4. *Express in degrees and radians, the angle whose circular measure is* $\frac{2\pi}{5}$.

The angle $= \frac{2\pi}{5}$ radians

Since π radians $= 180°$

$$\therefore \quad \frac{2\pi}{5} \text{ radians} = \frac{180}{\pi} \times \frac{2\pi}{5} = 72°$$

Also since π radians $= 200^g$

$$\therefore \quad \frac{2\pi}{5} \text{ radians} = \frac{200}{\pi} = 80^g.$$

5. *Express 4·4 radions in the (i) Sexagesimal (ii) Centesimal System.*

(*i*) π radians $= 180°$

$$1 \text{ radian} = \frac{180°}{\pi}$$

$$\therefore \quad 4{\cdot}4 \text{ radians} = \frac{180}{\pi} \times 4{\cdot}4 \text{ degrees}$$
$$= \frac{180}{22} \times 7 \times \frac{44}{10} \text{ degrees}$$
$$= 252°.$$

(*ii*) π radians $= 200^g$

$$1 \text{ radian} = \frac{200^g}{\pi}$$

$$\therefore \quad 4{\cdot}4 \text{ radians} = \frac{200}{\pi} \times 4{\cdot}4 \text{ grades}$$
$$= \frac{200}{22} \times 7 \times \frac{44}{10} \text{ grades}$$
$$= 280 \text{ grades.}$$

6. *Express in degrees as well as in radians the angle of a regular polygon of 16 sides.*

(*i*) sum of the exterior angles $= 360°$

$$\therefore \quad \text{each exterior angle} = \frac{360°}{16} = \frac{45°}{2}$$

$\therefore$ the interior angle $=180^\circ-\dfrac{45^\circ}{2}=\dfrac{315^\circ}{2}=157^\circ\ 30'$

(*ii*) $\because$ $180^\circ=\pi$ radians

$$\therefore\ \frac{315^\circ}{2}=\frac{315}{2}\times\frac{\pi}{180} \text{ radians}$$

$$=\frac{7\pi}{8} \text{ radians.}$$

7. *The angles of a triangle are in Arithmetical Progression. The number of grades in the least is to the number of radians in the greatest as 40 is to π. Find the angles in degrees.*

Let the angles be $(a-d)^\circ$, a°, $(a+d)^\circ$ (*P.U. 1942*)

Then $a-d+a+a+d=180^\circ$

[sum of the angles of a triangle].

$$\therefore\ a=60^\circ$$

$\therefore$ the angles are $(60-d)^\circ$, 60°, $(60+d)^\circ$

The least angle $=(60-d)^\circ$

$$=\frac{200}{180}\times(60-d). \quad [\because\ 180^\circ=200 \text{ grades}]$$

$$=\frac{10(60-d)}{9} \text{ grades}$$

The greatest angle $=(60+d)^\circ$

$$=\frac{\pi}{180}\times(60+d) \text{ radians}$$

$\therefore$ from the given data

$$\frac{\frac{10}{9}(60-d)}{\frac{\pi}{180}(60+d)}=\frac{40}{\pi}$$

$$\text{or}\quad \frac{200(60-d)}{60+d}=40$$

$$\text{or}\quad 5(60-d)=60+d$$

$$\text{or}\quad 300-5d=60+d$$

$$\therefore\quad d=40.$$

Hence the angles are 20°, 60°, 100°.

EXERCISE I (b)

1. Express in Circular Measure the angles.
$1^\circ, 1^g, 1', 1'', 1^{`}, 1^{``}$. *(C.U. 1881)*

2. Reduce 2·2 radians in the (i) Sexagesimal and (ii) Centesimal System.

3. Express in radians the fourth angle of a quadrilateral which has three angles of $46^\circ\ 31'\ 10''$, $75^\circ\ 44'\ 45''$, $123^\circ, 9'\ 35''$ respectively.

[Hint : Subtract the sum of all the three $\angle$s out of 360°, for all four $\angle$s of a quadrilateral $=360^\circ$. Then use $180=\pi$ radians.]

4. The vertical $\angle$ of an isosceles $\triangle$ is $55^\circ\ 17'$. Express base angles in Circular Measure. *(P.U. 1947)*

[Hint : Each base $\angle=\frac{1}{2}(180^\circ-55^\circ\ 17')$ etc.]

5. The difference of two $\angle$7s is 1° and Circular Measure of their sum is 1. Express each angle in grades. *(P.U. 1949)*

Hint : Let x° and y° be the two $\angle$s $\therefore\ x-y=1$. *(D. U. 1941)*

$$\text{and } (x+y)\frac{\pi}{180}=1$$

$$\therefore\quad x+y=\frac{180}{\pi}\text{ adding } 2x=\left(\frac{180}{\pi}+1\right)$$

$$\text{or}\quad x=\left(\frac{90}{\pi}+\frac{1}{2}\right)^\circ=\left(\frac{90}{\pi}+\frac{1}{2}\right)\frac{10^g}{9}=\left(\frac{100}{\pi}+\frac{5}{9}\right)^g$$

Similarly find 'y']

6. Find in degrees and radians, the angles of
   (i) a regular octagon *(P. U. 1936 Supp.)*
   (ii) a regular polygon of 40 sides *(P. U. 1944)*

[Hint : (i) a regular octagon has each exterior $\angle=\frac{360}{8}=45^\circ$.

($\because$ Sum of exterior $\angle$s$=360^\circ$.)

$\therefore$ each interior $\angle=(180-45^\circ)=135^\circ$

$$=\frac{134\times\pi}{180}=\frac{3}{4}\pi\text{ radians.}$$

(ii) Exterior $\angle$ of 40 sided polygon$=\frac{360^\circ}{40}=9^\circ$

$\therefore$ each interior $\angle=$......etc.]

7. If the angles of a $\triangle$ be in A. P. and one of them be 95°, find all of them in radians. (*P.U. 1931*)

[**Hint :** As in solved Ex. 7, suppose the $\angle$s to be $(a-d)^\circ$, a°, $(a+d)^\circ$.

Sum of $\angle s = a-d+a+a+d=110^\circ$.

$\therefore\ 3a=180^\circ$ or $a=60^\circ$.

Now $\angle 95^\circ$ is the greatest $\angle$, for a triangle cannot have two, obtuse $\angle$s.

$\therefore\ a+d=95^\circ$ or $60+d=95$ or $d=35$ etc]

8. Express in radians and degrees the angles of a triangle whose angles are in the ratio of 1 : 2 : 3 (*P. U.*)

9. In a rt $\angle$d $\triangle$, the difference between the two acute angles is $\frac{\pi}{9}$ in Circular Measure. Express the angles in degrees.

[**Hint :** Let one acute $\angle = x$ radians, the other $\angle = y$ radians.

$$\therefore\quad x-y=\frac{\pi}{9} \text{ and } x+y=\frac{\pi}{2}$$

$$\left[\because \text{ two acute } \angle s = 90^\circ\right]$$

Solve for x and y etc.

10. The angles of a triangle are in A.P., and the number of degrees in the least is to the number of radians in the greatest as $45 : \pi$. Find the angles in degrees.

[**Hint :** As in solved example 7.]

The Circular Measure of an angle, is the number of radians it contains and now we shall prove that :—

1·8. The Circular Measure of any angle at the centre of a circle $= \dfrac{\text{arc subtending that at the centre}}{\text{radius of the circle}}$

i.e. $\theta = \dfrac{l}{r}$ where l is the length of the arc, r is the radius of the circle and θ is the Circular Measure of the angle at centre. (*P.U. 1941, 48*)

Let O be centre of the circle and AOP be the angle subtended at the centre by the arc AP. Cut off arc AB equal to the radius, then $\angle$AOB $=1$ radian.

Let arc AB $=l$ and $\angle AOP=\theta$ (pronounced Theta).

Now $\dfrac{\angle AOP}{\angle AOB}=\dfrac{\text{arc AP}}{\text{arc AB}}=\dfrac{l}{r}$

[$\because$ $\angle$s at the centre are proportional to the arcs on which they stand.]

$\therefore \quad \dfrac{\theta \text{ radians}}{1 \text{ radian}}=\dfrac{l}{r}$

i.e., $\theta=\dfrac{l}{r}$, where θ is in Circular Measure or the measure of the angle at the centre in Circular units.

Cor. $\because \theta=\dfrac{l}{r} \quad \therefore \quad l=r\theta.$

Note. In the above relation there are three quantities. If any two of them are given, the third quantity can be found.

ILLUSTRATIVE EXAMPLES

1. *Find the number of degrees in the angles subtended at the centre of a circle of radius 10″ by an arc of length 33″.*

Here $r=10''$, $l=33''$.

Let θ be the number of radians in the angle

$\therefore \quad \theta=\dfrac{l}{r}=\dfrac{33}{10}$ radians

or $\quad \theta=\dfrac{33}{10}\times\dfrac{180^\circ}{\pi}$ [$\because$ π radians $=180^\circ$,

$=\dfrac{33}{10}\times\dfrac{180}{22}\times\dfrac{7}{1}=189^\circ.$

2. *An arc of a circle with centre O is 7·7 inches in length and the angle AOB is 2·2 radians. Calculate the radius of the circle.* **(P. U. 1949)**

$\theta=2\cdot2$ radians

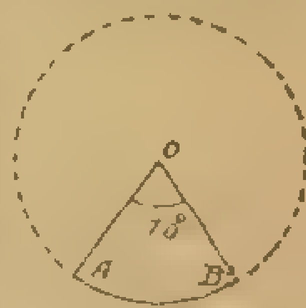
$l=7\cdot7''$

Use $\theta=\dfrac{l}{r} \quad \therefore \quad r=\dfrac{l}{\theta}=\dfrac{7\cdot7}{2\cdot2}=\dfrac{77}{22}=\dfrac{7}{2}=3\dfrac{1''}{2}.$

Here θ is already given in radians, hence we apply the formula directly.

3. *A horse is tethered to a stake by a rope 27 ft. long. If the horse moves along the circumference of a circle always keeping the rope tight, find how far it will have gone when the rope has traced out an angle of 70°.* (P. U. 1941)

Let O be the stake, and A the initial position of the horse. If the horse moves along the circumference, let B be its final position, when the angle AOB described $=70°$, and the radius of the circle, of which AB is an arc, shall be equal to the length of the rope.



Now $\theta=\frac{l}{r}$, where $\theta=70°=70\times\frac{\pi}{180}$ radians

$$=\frac{70\pi}{180}\text{ radians.}$$

and $r=27'$.

$\therefore$ arc $AB=l=r\theta=27\times\frac{70\pi}{180}\left(\text{putting } \pi=\frac{22}{7}\right)$

$$=\frac{27\times70}{180}\times\frac{22}{7}=33\text{ ft.}$$

Note. Whenever the *value* of l is required, we should write the formula $\theta=\frac{l}{r}$ as $l=r\theta$, after cross multiplication.

4. *The Moon's distance from the Earth is 240,000 miles and its diameter subtends an angle of 31′ at the eye of the observer. Find the diameter of the Moon.* (P. U.)

Let AB be the diameter of the Moon, E the eye of the observer ; $\angle AEB=31'$.

$\because$ $\angle AEB$ is *small* $\therefore$ AB is approximately equal to the small arc of a circle whose centre is E and radius is equal to the distance of the Moon from the Earth. Let $AB=l$ miles.

θ = number of radians in $31'$

$= \dfrac{31}{60} \times \dfrac{\pi}{180}$

$r = 240{,}000$ miles.

Since $l = r\theta$ $\left(\because \theta = \dfrac{l}{r}\right)$

$\therefore \quad l = 240{,}000 \times \dfrac{31}{60} \times \dfrac{\pi}{180}$ miles

$= 240000 \times \dfrac{31}{60} \times \dfrac{22}{7} \times \dfrac{1}{180}$ miles

$= \dfrac{6200}{9} \times \dfrac{22}{7} = \dfrac{136400}{63}$ miles $= 2165$ miles nearly.

5. *Assuming the average distance of Earth from the Sun to be 93,000,000 miles and the angle subtended by the Sun at the eye of the observer on the Earth to be 32′, find the Sun's diameter.* See Fig 5. (*P.U. 1948*)

Let D be the diameter of the Sun in miles.

The angle subtended by the Sun being very small, its diameter may be taken approximately to be a small arc of a circle whose centre is the eye of the person and radius = distance of the Sun from the Earth.

θ = the number of radians in $32'$

$= \dfrac{32}{60} \times \dfrac{\pi}{180} = \dfrac{2\pi}{675}$

$r = 93{,}000{,}000$ miles

$\therefore$ Putting the values of θ, l, r in the formula $\theta = \dfrac{l}{r}$,

we get $\dfrac{2\pi}{675} = \dfrac{D}{93000000}$

$\therefore \quad D = \dfrac{186000000}{675}\pi$ miles

$= \dfrac{186000000}{675} \times \dfrac{22}{7}$ miles $= 866{,}000$ miles approx.

EXERCISE I (c)

1. Find the number of degrees in the angle subtended at the centre of a circle of radius 10 by an arc of length 20.

(*P.U.*)

[**Hint** : $r=10$, $l=20$ $\therefore$ $\theta=\dfrac{l}{r}=2$ radians.

Convert radians into degrees by formula $\pi=180°$ and reduce it to minutes and seconds also.]

2. The large hand of a clock is 2 ft, 4 inches long. How many inches does its extremity move in 20 minutes ?

(P.U. 1941, Supp.)

[**Hint** : $\angle$moved in 20 minutes$=120°=\dfrac{2\pi}{3}$ radians $r=28''$; find l.

3. A circular wire of radius 3″ is cut and bent so as to lie along the circumference of a hoop whose radius is 4′. Find in grades the angle which is subtended at the centre of the hoop.

(P. U. 1938)

[**Hint** : Length of the arc placed at the $\odot^{ce}$ of the hoop $=\odot^{ce}$ of the circle of radius 3″

$$=2\pi\times\frac{3}{12}=\frac{\pi}{2}\text{ ft.} \qquad [\because \quad \odot^{ce}=2\pi r]$$

$r=4$ ft. apply $\theta=\dfrac{l}{r}$ to find θ.]

4. A railway train is travelling on a curve of half a mile radius at the rate of 20 miles per hour ; through what angle has it turned in 10 seconds $\left(\pi=\dfrac{22}{7}\right)$

(P.U. 1941 Supp.)

[**Hint** : Length of arc$=$distance described in 10 seconds at the given rate of 20 miles per hour.]

5. At what distance does a man whose height is 6 ft. subtended an angle of 10′ ? *(M. U.)*

[**Hint** : As the angle is very small, the height of the man shall approximately = the arc of the circle, whose radius shall be the required distance.]

6. Assuming that the Earth's radius is 3960 miles, and that it subtends an angle of 57′ at the centre of the Moon, find the distance of the Moon from the Earth's centre. *(P.U.)*

7. Find the distance from the eye at which a pice (diameter one inch) should be held so as to conceal the full Moon whose angular diameter is 31′.

[Hint: The moon will not be visible if the diameter of the pice also subtends an $\angle 31'$ at the eye.]

8. How long does it take the hour hand of a clock to rotate through an angle whose circular measure $\frac{\pi}{30}$? (*P.U.*)

MISCELLANEOUS EXERCISES ON CHAPTER I

1. What is the (*i*) Sexegesimal System, (*ii*) Centesimal System and (*iii*) Circular System? (*P.U. 1944*)

2. In what quardrant does the final position of the revolving line lie, if it has traced out an angle of

$$(i)\ 1320^\circ \quad (ii)\ 980^g \quad (iii)\ \frac{17\pi}{3} \text{ radians ?}$$

3. From an angle of 60°, 30 grades are taken; what is the Circular Measure of the remainder? (*P.U.*)

4. The Circular Measure of one angle of a triangle is $\frac{3\pi}{10}$, the second is 70 grades; find the third in degrees.

(*C. U. 1874*)

5. If the number of sides of two regular polygons be m and n respectively and the number of grades in the angle of the first polygon be equal to the number of degrees in the angle of the second polygon, prove that

$$\frac{20}{m}=\frac{18}{n}+1.$$

[Hint: Sum of angles of a regular polygon $=(2n-4)$ right angles, where n denotes the number of sides

$$\therefore \text{ each } \angle \text{ of the first} = \left(\frac{2m-4}{m}\right) \text{right } \angle\text{s}$$

$$=\frac{2m-4}{m}\times 100^g$$

$$\text{and each } \angle \text{ of the second} = \frac{2n-4}{n} \text{ right } \angle\text{s}$$

$$=\frac{2n-4}{n}\times 90^\circ, \text{ etc.]}$$

6. The angles of a quadrilateral are in A.P., and the greatest angle is double the least angle. Express the least angle in radians.

[Hint : When four things are in A.P., we rssume the four things as $a-3d$, $a-d$, $a+d$, $a+3d$, etc.]

7. If D, G, C, are respectively the bumber of degrees, grades and radians in an angle, show that

$$\frac{D}{90}=\frac{G}{100}=\frac{2C}{\pi}$$

Sol : Suppose the angle contains x rt. $\angle$s,

$\therefore \quad x=\frac{D}{90}$ rt. $\angle$s ($\because$ $90^\circ=1$ rt. $\angle$ and there are D° in the $\angle$)

Also $x=\frac{G}{100}$ rt. $\angle$s ($\because 100^g=1$ rt. $\angle$ and ,, G^g ,, $\angle$)

Also $x=\dfrac{C}{\frac{\pi}{2}}$ rt. $\angle$s ($\because$ $\pi^c=2$ rt. $\angle$s and ,, C radians ,, $\angle$)

As each of these ratios is equal to x $\therefore$ they are also equal.]

8. If G, D, and θ be the number of grades, degrees and radians in any angle, prove that

$$G-D=\frac{20\theta}{\pi}$$ (P.U.)

[Hint : Let the angle contain x rt. $\angle$s.

$\therefore$ $G=100x$ and $D=90x$ and $\theta=\frac{\pi}{2}x$.

$\therefore$ $G-D=100x-90x=10x$. Put the value of x in terms of θ from the third relation.]

9. The perimeter of a certain sector of a circle is equal to half that of the circle of which it is a sector. Find the Circular Measure of the angle of the sector. (*P. U. 1914 Supp.*)

[Hint : Perimeter of a sector $=2r+$length of the arc
$=2r+r\theta$ (where θ is the angle of the sector)
$=\frac{1}{2}\times 2\pi r$ (given) etc.]

10. If the diameter of the Moon subtends an angle af $30'$ at the eye of the observer, and the diameter of the Sun an angle of 32 and if the distance of the Sun be 375 times the distance of the Moon, find the ratio of their diameters.

11. What is the difference of latitude of two places, one of which is due North of the other, at a distance of 60 miles from it, the radius of the Earth being 400 miles ?

[**Hint** : Latitude is the angular distance on a meridian, *i.e.*, angle subtended at centre of the Earth].

12. There are three angles ; the Circular Measure of the first exceeds that of the second by $\frac{\pi}{10}$, the sum of the second and the third is 30 grades, and the sum of the first and the second is 35 degrees. Determine the three angles

ANSWERS TO EXERCISES IN CHAPTER I

Exercises of Art. 1·3. **1.** (*i*) 1st quadrant.
(*ii*) 4th quadrant.

Exercise I (*a*) **1.** $26^g\ 5'\ 79{\cdot}5''$. **2.** $23^\circ\ 28'\ 45''$.
4. $47\frac{7^\circ}{19}$, $42\frac{12}{19}$. **7.** 45°, 27°. **8** 35°, 25°.

Exercise I (*b*) **1.** $\frac{\pi}{180}$, $\frac{\pi}{200}$, $\frac{\pi}{180\times60}$, $\frac{\pi}{180\times60\times60}$.
$\frac{\pi}{200\times100}$, $\frac{\pi}{200\times100\times100}$.
2. 126°, 140^g. **3.** $\frac{13751}{21600}\pi$. **4.** $\frac{7483}{21600}\pi$.
5. $\left(\frac{100}{\pi}+\frac{5}{9}\right)^g$, $\left(\frac{100}{\pi}-\frac{5}{9}\right)^g$. **6.** 171°, $\frac{19}{20}\pi$.
7. $\frac{5\pi}{36}$, $\frac{\pi}{3}$, $\frac{19\pi}{36}$. **8.** $\frac{\pi}{6}$, $\frac{\pi}{3}$, $\frac{\pi}{2}$; 30°, 60°. 90°
9. 55°, 35°. **10.** 24°, 60°, 96°.

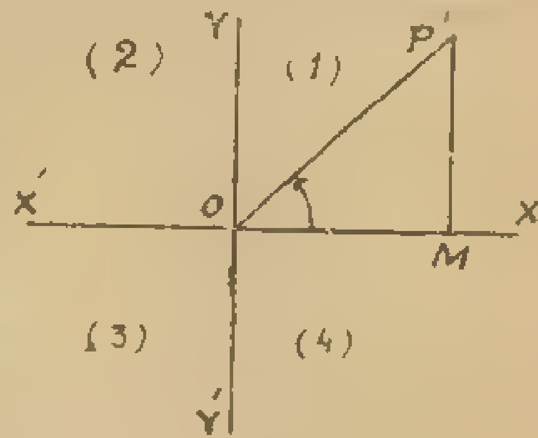
Exercise I (*c*) **1.** $114^\circ\ 35'\ 29{\cdot}6''$ (by taking $\pi=\frac{355}{113}$)
2. $58\frac{2}{3}$ inches. **3** 25^g.
4. $6\frac{4}{11}^\circ$ (approximately).
5 2062·65 ft. nearly. **6.** 238·737 inches.
7. 9·2 ft. nearly. **8.** 1 hr. 12 minutes.

Miscellaneous Exercises (Chap. I)
2. (*i*) third ; (*ii*) second ; (*iii*) fourth.
3. $\frac{11\pi}{60}$. **4.** 63°. **6.** $\frac{\pi}{3}$. **9.** $\pi-2$.
10. 1 : 400. **11.** $51'\ 34'$. **12.** 27°, 9°, 18°.

CHAPTER II

TRIGONOMETRICAL RATIOS

2·1. Let XOX′ and YOY′ be two lines at right-angles intersecting at O. The plane is then divided into four quadrants as shown in the figure. The lines XOX′ and YOY′ are known as the axes of x and y respectively. O is called the origin.

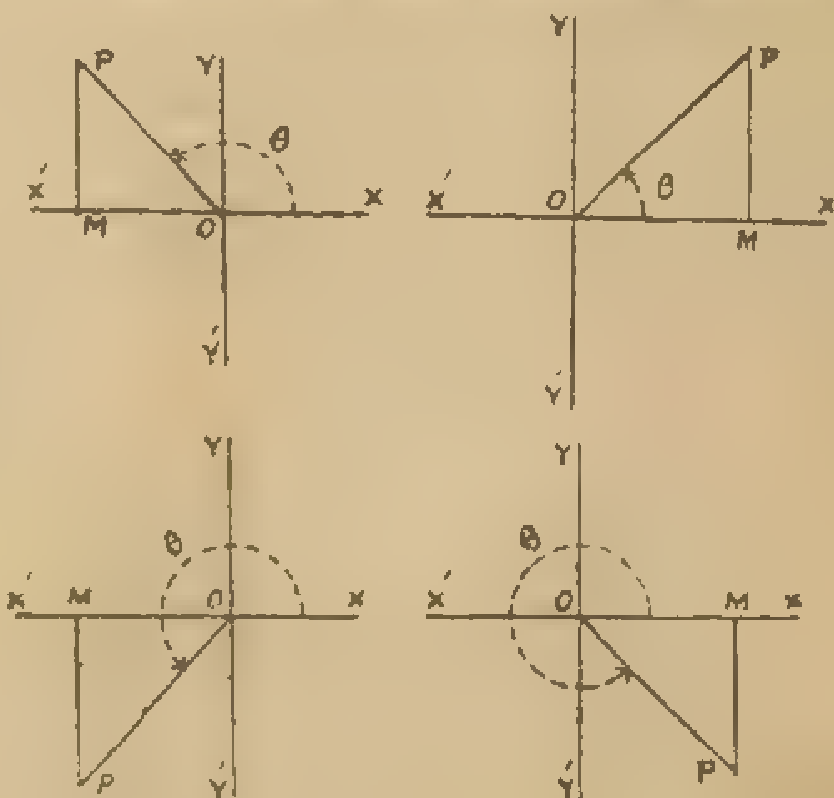


Let OP, the revolving line, trace out an angle θ (pronounced as theta). Take P any pt. on the line OP and draw PM ⊥ to the line XOX′. Then OM is taken to be positive if M is to the right of the origin O and OM is taken to be negative if M is to the left of the origin Also MP is positive if above the line X′OX and is taken as negative if it lies below the line X′OX. But the line OP is always taken as +ve.

Note: Remember *plus to the right*; *minus to the left*; *positive height*; *negative depth*.

2·11. To define the trigonometrical ratios of angles of any magnitude.

Lef XOX′ and YOY′ be the axis of x and y respectively. Let the revolving line OP starting from OX, trace out any angle XOP $=\theta$. Draw PM perpendicular to X′OX′, so that ∠XOP $=\theta$.



The figure shows the positions of OP in the four quadrants. With due regard to the signs of the lines

(1) $\frac{MP}{OP}$ is called the *sine* of θ (writteg as *sin* θ)

(*P.U. 1938, 1940*)

(2) $\frac{OM}{OP}$ is called the *cosine* of θ (written as *cos* θ)

(3) $\frac{MP}{OM}$ is called the *tangent* of θ (written as *tan* θ)

(*P.U. 1940, 1943*)

(4) $\frac{OM}{MP}$ is called the *co tangent* of θ (written as *cot* θ)

(5) $\frac{OP}{OM}$ is called the *secant* of θ (written as *sec* θ)

(*P. U. 1939*)

(6) $\frac{OP}{MP}$ is called the *cosecant* of θ (written as cosec θ).

Observe that cot θ is the reciprocal of tan θ, sec θ is the reciprocal of cos θ and cosec θ is the reciprocal of sin θ.

Note 1. The above six ratios are called the trigonometrical ratios of θ. They are also called the *trigonometrical* or *circular functions* of θ.

Note 2. The circular functions being the ratios of lengths are all *pure numbers*.

Note 3. In addition to the above six ratios, there are two more ratios but they are not very important, and seldom used.

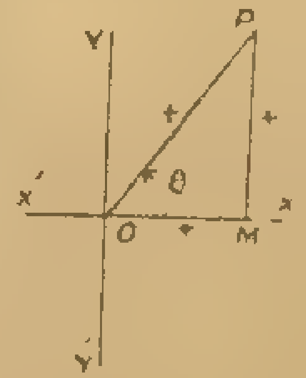
(7) $1-\cos\theta$ or $1-\frac{OM}{OP}$ is called the *versed sine of* θ

(written as versed θ)

and (8) $1-\sin\theta$ or $1-\frac{MP}{OP}$ is called the *coversed sine of* θ (written as covers θ). These two are very seldom used.

2·2. To find the signs of the trigonometrical ratios in different quadrants.

Let the revolving line start from OX and trace out an angle $XOP=\theta$ in different quadrants.



Draw $PM \perp X'OX$

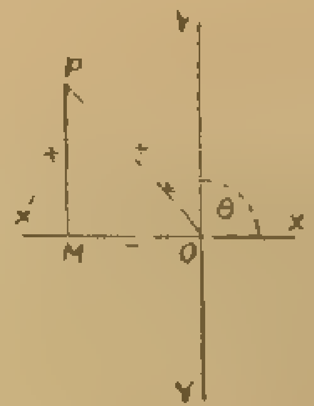
(1) *In the first quadrant*

all the lines are +ve (positive)

$\therefore$ all the ratios are positive

(2) *In the second quadrant*

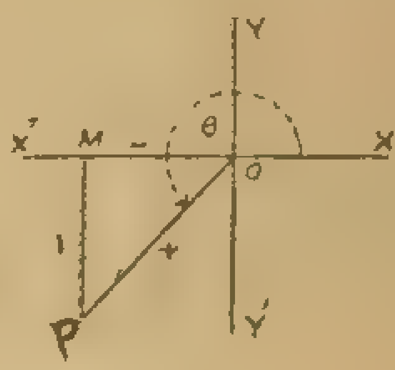
OM is −ve (negative), MP is +ve and OP is +ve



$\therefore$ sin θ is +ve
cos θ is −ve
tan θ is −ve
cot θ is −ve
sec θ is −ve
cosec θ is +ve

(3) *In the third quadrant*

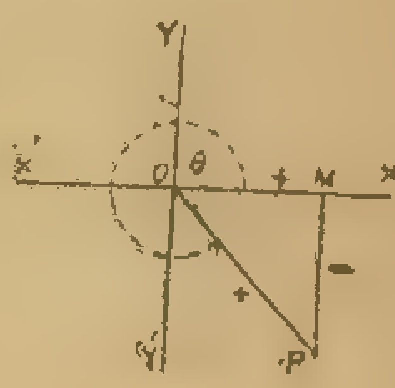
OM and MP are −ve, OP is +ve



$\therefore$ sin θ is −ve
cos θ is −ve
tan θ is +ve
cot θ is +ve
sec θ is −ve
cosec θ is −ve

(4) *In the fourth quadrant*

MP is −ve, OM is +ve, OP is +ve



$\therefore$ sin θ is −ve
cos θ is +ve
tan θ is −ve
cot θ is −ve
sec θ is +ve
cosec θ is −ve

Note. These results may be summarised as follows :

Quadrant II	Quadrant I
sin + cosec +	all +
tan + cot +	cos + sec +
Quadrant III	Quadrant IV

EXERCISES

1. In what quadrant can θ lie if

(i) $\sin\theta=-\frac{3}{4}$; (ii) $\cos\theta=\frac{1}{2}$; (iii) $\tan\theta=-\sqrt{3}$; (iv) $\sec\theta=\sqrt{2}$; (v) $\text{cosec}\,\theta=-2$?

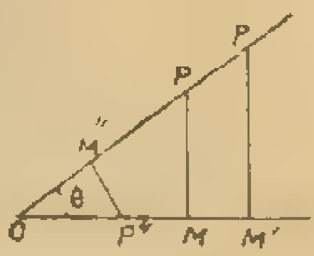
2. State which ratios for the following angles are positive or negative :

(i) $400°$, (ii) $-160°$, (iii) $\dfrac{-10\pi}{3}$.

[**Hint :** Find in what quadrant will the revolving line lie when it has traced out the above angles]

2·3. To prove that the trigonometrical ratios are always the same for the same angle.

Let the revolving line OA start from OX and trace out angle XOA$=\theta$. PM and P$'$M$'$ are perpendicular to OX from any two points P and P$'$ in OA ; P$''$M$''$ is perpendicular to OA from any point P$''$ in OX.



The triangles OPM, OP$'$M$'$ and OM$''$P$''$ are equiangular and therefore similar ;

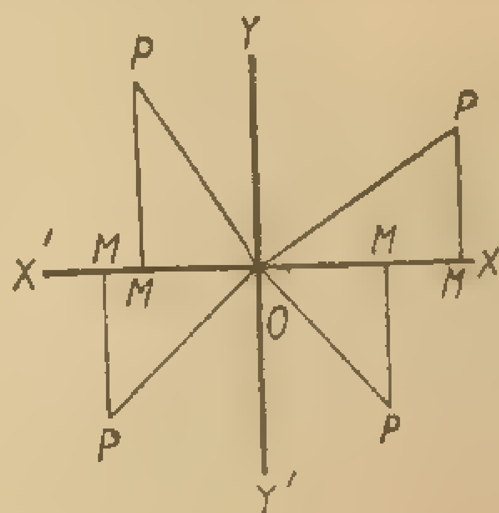
$$\therefore\ \frac{MP}{OP}=\frac{M'P'}{OP'}=\frac{P''M''}{OP''}=\sin\theta.$$

$\therefore$ Sine of an angle is the same whatever the triangle of reference. In a like manner it can be shown that all trigonometrical ratios do not depend on the size of the triangle. The ratios are dependent on the magnitude of the angle only.

2·4. The positive integral powers of t-ratios are denoted thus :—$(\sin\theta)^2$ is denoted by $\sin^2\theta$ and is read as 'sin square θ' ; $(\sin\theta)^3$ is denoted by $\sin^3\theta$ and is read as 'sin cubed θ'. And so on for the other ratios.

But be careful that $(\sin\theta)^{-1}$ is never written as $\sin^{-1}\theta$ and so for the other ratios. The latter notation has got a different meaning.

2·5. Fundamental relations between the different trigonometrical ratios of an angle.



Let a revolving line start from OX and trace out any angle $XOP=\theta$.

Draw $PM \perp X'OX$. Then

(1) $\sin\theta \times \operatorname{cosec}\theta$

$$=\frac{MP}{OP}\times\frac{OP}{MP}=1$$

$$\therefore \operatorname{cosec}\theta=\frac{1}{\sin\theta}$$

$$\text{or}\quad \sin\theta=\frac{1}{\operatorname{cosec}\theta}$$

$$(2)\ \cos\theta\ \sec\theta=\frac{OM}{OP}\cdot\frac{OP}{OM}=1$$

$$\therefore \sec\theta=\frac{1}{\cos\theta}$$

$$\text{or}\quad \cos\theta=\frac{1}{\sec\theta}.$$

$$(3)\ \tan\theta\ \cot\theta=\frac{MP}{OM}\cdot\frac{OM}{MP}=1$$

$$\therefore \cot\theta=\frac{1}{\tan\theta}$$

$$\text{or}\quad \tan\theta=\frac{1}{\cot\theta}.$$

(4) $$\frac{\sin\theta}{\cos\theta}=\frac{\frac{MP}{OP}}{\frac{OM}{OP}}=\frac{MP}{OP}\times\frac{OP}{OM}=\frac{MP}{OM}=\tan\theta$$

$$\therefore \quad \tan\theta=\frac{\sin\theta}{\cos\theta}.$$

(5) $$\frac{\cos\theta}{\sin\theta}=\frac{\frac{OM}{OP}}{\frac{MP}{OP}}=\frac{OM}{OP}\times\frac{OP}{MP}=\frac{OM}{MP}=\cot\theta$$

$$\therefore \quad \cot\theta=\frac{\cos\theta}{\sin\theta}.$$

(6) For all values of θ, we have from the rt.-angled $\triangle$ OMP,

$$MP^2+OM^2=OP^2 \qquad \ldots \quad (1)$$

(*a*) Dividing (1) by OP^2 we get

$$\left(\frac{MP}{OP}\right)^2+\left(\frac{OM}{OP}\right)^2=1$$

or $$(\sin\theta)^2+(\cos\theta)^2=1$$

or $$\sin^2\theta+\cos^2\theta=1. \qquad (P.\ U.\ 1938)$$

Cor. 1. $\sin^2\theta=1-\cos^2\theta.$

Cor. 2. $\cos^2\theta=1-\sin^2\theta.$

(*b*) Dividing (1) by OM^2, we have

$$\left(\frac{MP}{OM}\right)^2+1=\left(\frac{OP}{OM}\right)^2$$

or $$\tan^2\theta+1=\sec^2\theta$$

or $$\sec^2\theta=1+\tan^2\theta. \qquad (P\ U.\ 1945,\ 47)$$

Cor. 1. $\tan^2\theta=\sec^2\theta-1.$

Cor. 2. $\sec^2\theta-\tan^2\theta=1.$

(*c*) Dividing (1) by MP^2, we have

$$1+\left(\frac{OM}{PM}\right)^2=\left(\frac{OP}{MP}\right)^2$$

i.e., $$1+\cot^2\theta=\operatorname{cosec}^2\theta$$

or $$\operatorname{cosec}^2\theta=1+\cot^2\theta.$$

Cor. 1. $\cot^2\theta=\operatorname{cosec}^2\theta-1.$

Cor. 2. $\operatorname{cosec}^2\theta-\cot^2\theta=1.$

Note. The above six formulæ are very important. The first three are known as reciprocal relations, fourth and fifth are known as quotient relations and sixth ('*a*', '*b*', '*c*',) are known as square relations.

Trigonometric Identities

When two given expressions involving the trigonometric functions of an angle have to be proved equal, this can be effected by treating the more complicated side of the identity, first by means of the above six formulæ and then reducing it to the other side. The examples given below illustrate the method.

Example 1. *Prove that* $cosec^2\theta+sec^2\theta=cosec^2\theta\ sec^2\theta$ (*P.U.*)

$$\text{Left hand side}=\text{cosec}^2\theta+\sec^2\theta$$
$$=\frac{1}{\sin^2\theta}+\frac{1}{\cos^2\theta}$$
$$=\frac{\cos^2\theta+\sin^2\theta}{\sin^2\theta\ \cos^2\theta}=\frac{1}{\sin^2\theta\ \cos^2\theta}$$
$$=\text{cosec}^2\theta.\ \sec^2\theta=\text{Right-hand side.}$$

Example 2. *Prove that* $(tan\ \theta+sec\ \theta)^2=\frac{1+sin\ \theta}{1-sin\ \theta}$ (*C.U.* 1908)

$$\text{L.H.S.}=\left(\frac{\sin\ \theta}{\cos\ \theta}+\frac{1}{\cos\ \theta}\right)^2$$
$$=\frac{(1+\sin\ \theta)^2}{\cos^2\theta}$$
$$=\frac{(1+\sin\ \theta)^2}{1-\sin^2\theta}\ (\because\ \sin^2\theta+\cos^2\theta=1)$$
$$=\frac{(1+\sin\ \theta)^2}{(1+\sin\ \theta)(1-\sin\ \theta)}\ [\text{since } a^2-b^2=(a+b)(a-b)]$$
$$=\frac{1+\sin\ \theta}{1-\sin\ \theta}=\text{R.H.S.}$$

Example 3. *Prove that* $sin\ \alpha(1+tan\ \alpha)+cos\ \alpha(1+cot\ \alpha)$ $=sec\ \alpha+cosec\ \alpha.$

$$\text{L.H.S.}=\sin\ \alpha\left(1+\frac{\sin\ \alpha}{\cos\ \alpha}\right)+\cos\ \alpha\left(1+\frac{\cos\ \alpha}{\sin\ \alpha}\right)$$
$$=\sin\ \alpha\left(\frac{\cos\ \alpha+\sin\ \alpha}{\cos\ \alpha}\right)+\cos\ \alpha\left(\frac{\sin\ \alpha+\sin\ \alpha}{\sin\ \alpha}\right)$$

$$=(\sin\alpha+\cos\alpha)\left(\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}\right)$$

$$=(\sin\alpha+\cos\alpha)\left(\frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha\cos\alpha}\right)$$

$$=\frac{(\sin\alpha+\sin\alpha)}{\sin\alpha\cos\alpha}=\frac{\sin\alpha}{\sin\alpha\cos\alpha}+\frac{\cos\alpha}{\sin\alpha\cos\alpha}$$

$$=\frac{1}{\cos\alpha}+\frac{1}{\sin\alpha}=\sec\alpha+\text{cosec}\,\alpha=\text{R.H.S.}$$

Example 4. (*a*) *Show that* $\text{cosec}^2 A-1=\cos^2 A\,\text{cosec}^2 A.$ (*P.U. 1922*)

$$\text{L.H.S.}=\text{cosec}^2 A-1=\frac{1}{\sin^2 A}-1$$

$$=\frac{1-\sin^2 A}{\sin^2 A}=\frac{\cos^2 A}{\sin^2 A}$$

$$=\cos^2 A\,\text{cosec}^2 A=\text{R.H.S.}$$

Example 5. *Show that* $(\sec\phi-\cos\phi)(\text{cosec}\,\phi-\sin\phi)$

$$=\frac{\tan\phi}{1+\tan^2\phi}$$ (*P.U. 1950*)

$$\text{L.H.S.}=\left(\frac{1}{\cos\phi}-\cos\phi\right)\left(\frac{1}{\sin\phi}-\sin\phi\right)$$

$$=\frac{1-\cos^2\phi}{\cos\phi}\cdot\frac{1-\sin^2\phi}{\sin\phi}$$

$$=\frac{\sin^2\phi}{\cos\phi}\cdot\frac{\cos^2\phi}{\sin\phi}=\sin\phi\cdot\cos\phi$$

$$\text{R.H.S.}=\frac{\dfrac{\sin\phi}{\cos\phi}}{1+\dfrac{\sin^2\phi}{\cos^2\phi}}=\frac{\sin\phi\cdot\cos\phi}{\cos^2\phi+\sin^2\phi}=\frac{\sin\phi\cos\phi}{1}=\sin\phi.\cos\phi$$

$\therefore$ L.H.S.=R.H.S.

(*b*) $(\text{cosec}\,A-\sin A)(\sec A-\cos A)(\tan A+\cot A)=1.$

$$\text{L.H.S.}=\left(\frac{1}{\sin A}-\sin A\right)\left(\frac{1}{\cos A}-\cos A\right)\left(\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}\right)$$

$$=\frac{1-\sin^2 A}{\sin A}\times\frac{1-\cos^2 A}{\cos A}\times\frac{\sin^2 A+\cos^2 A}{\cos A\sin A}$$

$$=\frac{\cos^3 A}{\sin A}\times\frac{\sin^3 A}{\cos A}\times\frac{1}{\sin A\cos A}$$

$$=\frac{\sin^2 A\cos^2 A}{\sin^2 A\cos^2 A}=1=\text{R.H.S.}$$

Example 6. *Prove that* $\sqrt{\dfrac{1+\cos A}{1-\cos A}}=\operatorname{cosec} A+\cot A.$ *(C. U. 1937)*

$$\text{L.H.S.}=\sqrt{\frac{1+\cos A}{1-\cos A}}=\sqrt{\frac{(1+\cos A)(1+\cos A)}{(1-\cos A)(1+\cos A)}}$$

$$=\sqrt{\frac{(1+\cos A)^2}{1-\cos^2 A}}=\frac{1+\cos A}{\sin A}$$

$$=\frac{1}{\sin A}+\frac{\cos A}{\sin A}$$

$$=\operatorname{cosec} A+\cot A=\text{R.H.S.}$$

Example 7. *Prove that*

$$\frac{1}{\operatorname{cosec} A-\cot A}-\frac{1}{\sin A}=\frac{1}{\sin A}-\frac{1}{\operatorname{cosec} A+\cot A}$$

$$\text{L.H.S.}=\frac{(\operatorname{cosec} A+\cot A)}{(\operatorname{cosec} A-\cot A)(\operatorname{cosec} A+\cot A)}-\frac{1}{\sin A}$$

$$=\frac{\operatorname{cosec} A+\cot A}{\operatorname{cosec}^2 A-\cot^2 A}-\operatorname{cosec} A$$

$$(\because \operatorname{cosec}^2 A-\cot^2 A=1)$$

$$=\operatorname{cosec} A+\cot A-\operatorname{cosec} A=\cot A.$$

$$\text{R.H.S.}=\frac{1}{\sin A}-\frac{\operatorname{cosec} A-\cot A}{(\operatorname{cosec} A+\cot A)(\operatorname{cosec} A-\cot A)}$$

$$=\operatorname{cosec} A-\frac{(\operatorname{cosec} A-\cot A)}{(\operatorname{cosec}^2 A-\cot^2 A)}$$

$$=\operatorname{cosec} A-\operatorname{cosec} A+\cot A=\cot A.$$

$\therefore$ L.H.S. = R.H.S.

Example 8. *If* $u_n=\cos^n\theta+\sin^n\theta$ *; show that*

$$2u_6-3u_4+1=0.$$ *(P. U. 1938)*

Put $n=4$, then $u_4=\cos^4\theta+\sin^4\theta$

Again put $n=6$, then $u_6=\cos^6\theta+\sin^6\theta$

Now the question is to show that

$$2(\cos^6\theta+\sin^6\theta)-3(\cos^4\theta+\sin^4\theta)+1=0.$$

$$\text{L.H.S.}=2(\cos^2\theta+\sin^2\theta)(\cos^4\theta-\sin^2\theta\cos^2\theta+\sin^4\theta)-3\cos^4\theta-3\sin^4\theta+1$$

$$[\because\ a^3+b^3=(a+b)(a^2-ab+b^2)]$$

$$=2(\cos^4\theta-\sin^2\theta\cos^2\theta+\sin^4\theta)-3\cos^4\theta-3\sin^4\theta+1$$

$$=2\cos^4\theta-2\sin^2\theta\cos^2\theta+2\sin^4\theta-3\cos^4\theta-3\sin^4\theta+1$$

$$=1-\cos^4\theta-2\sin^2\theta\cos^2\theta-\sin^4\theta$$

$$=1-(\cos^4\theta+2\sin^2\theta\cos^2\theta+\sin^4\theta)$$

$$=1-(\cos^2\theta+\sin^2\theta)^2$$

$$=1-1=0=\text{R.H.S.}$$

Example 9. *Prove that*

$$\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}=\sec A\ \text{cosec}\ A+1$$

$$\text{The L.H.S.}=\frac{\dfrac{\sin A}{\cos A}}{1-\dfrac{\cos A}{\sin A}}+\frac{\dfrac{\cos A}{\sin A}}{1-\dfrac{\sin A}{\cos A}}$$

$$=\frac{\dfrac{\sin A}{\cos A}}{\dfrac{\sin A-\cos A}{\sin A}}+\frac{\dfrac{\cos A}{\sin A}}{\dfrac{\cos A-\sin A}{\cos A}}$$

$$=\frac{\sin^2 A}{\cos A(\sin A-\cos A)}+\frac{\cos^2 A}{\sin A(\cos A-\sin A)}$$

$$=\frac{\sin^2 A}{\cos A(\sin A-\cos A)}-\frac{\cos^2 A}{\sin A(\sin A-\cos A)}$$

$$=\frac{\sin^3 A-\cos^3 A}{\sin A\cos A(\sin A-\cos A)}$$

$$=\frac{(\sin A-\cos A)(\sin^2 A+\sin A\cos A+\cos^2 A)}{\sin A\cos A(\sin A-\cos A)}$$

$$=\frac{1+\sin A\cos A}{\sin A\cos A}=\frac{1}{\sin A\cos A}+\frac{\sin A\cos A}{\sin A\cos A}$$

$$=\sec A\ \text{cosec}\ A+1=\text{R. H. S.}$$

EXERCISE II (a)

Prove the following identities :—(1 – 25).

1. $\sec^2 A-\tan^2 A-1=0.$ *(P. U. 1921)*
2. $(1+\tan^2 A)(1-\sin^2 A)=1.$ *(C. U. 1906)*

3. $\dfrac{\text{cosec } \theta}{\sec \theta}+\dfrac{\sec \theta}{\text{cosec } \theta}=\sec \theta \text{ cosec } \theta$

[**Hint:** Change the L.H.S. into sines and cosines and simplify.]

4. (*i*) $\dfrac{1}{\cot A+\tan A}=\sin A \cos A.$

(*ii*) $\dfrac{1}{\sec A-\tan A}=\sec A+\tan A.$

[**Hint:** Put $1=\sec^2 A-\tan^2 A$ or cross multiply.]

5. (*i*) $\cos^4\theta-\sin^4\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1.$ (*P. U.*)

(*ii*) $\tan^2 A-\sin^2 A=\sin^4 A \sec^2 A.$

6. (*i*) $\dfrac{1-\tan \theta}{1+\tan \theta}=\dfrac{\cot \theta-1}{\cot \theta+1}.$

[**Hint:** Put $\tan \theta=\dfrac{1}{\cot \theta}$ in the L H.S.]

(*ii*) $\dfrac{\cot \theta+\tan \theta}{\text{cosec } \theta}=\cos\theta.$

7. $(1+\sin A+\cos A)^2=2(1+\sin A)(1+\cos A).$

8. $\dfrac{\sin A}{1+\cos A}=\dfrac{1-\cos A}{\sin A}.$ [**Hint:** Cross multiply.]

9. $1-\tan^4 A=2\sec^2 A-\sec^4 A.$

[**Hint:** $\text{L. H. S.}=(1+\tan^2 A)(1-\tan^2 A)$

$=\sec^2 A[1-(\sec^2 A-1)].$

10. $\dfrac{\sec A-\tan A}{\sec A+\tan A}=1-2\sec A \tan A+2\tan^2 A.$

Hint: Multiply the numerator and denominator of L.H.S. by (sec A−tan A.]

11. $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1}=\dfrac{1+\sin A}{\cos A}.$

[**Hint:** Put R.H.S. as sec A+tan A and then cross multiply.]

12. $\sec^2\theta+\text{cosec}^2\theta=(\tan \theta+\cot \theta)^2.$ (*D. U. 1940*)

13. $\dfrac{1}{\sec \theta+\tan \theta}=\dfrac{1-\sin \theta}{\cos \theta}.$ (*P. U. 1945*)

14. (*i*) $(1+\sec A+\tan A)(1-\sec A+\tan A)=2\tan A$.

SoLe (*ii*) $(1+\cot A-\operatorname{cosec} A)(1+\tan A+\sec A)=2$.

(*D. U. 1940*)

Hint : (*i*) L. H. S. $=(1+\tan A)^2-\sec^2 A$, simplify.

15. (*i*) $\sqrt{\dfrac{1-\cos A}{1+\cos A}}=\operatorname{cosec} A-\cot A$

(*ii*) $\sqrt{\dfrac{1-\sin A}{1+\sin A}}=\sec A-\tan A$. (*P. U. 1941-S*)

[Hint : (*i*) Multiply the numerator and denominator of L.H.S. by $1-\cos A$ and simplify.]

16. (*i*) $(\sin A+\cos A)(1-\sin A\cos A)=\sin^3 A+\cos^3 A$.

[Hint : From factors of R. H. S.]

(*ii*) $\sec^2\theta\operatorname{cosec}^2\theta=\tan^2\theta+\cot^2\theta+2$.

[Hint : Change $\sec^2\theta$ and $\operatorname{cosec}^2\theta$ into $1+\tan^2\theta$ and $1+\cot^2\theta$ and multiply out.]

17. $\sec^6 A=\tan^6 A+3\tan^2 A\sec^2 A+1$. (*P. U.*)

18. $(\sin\alpha+\operatorname{cosec}\alpha)^2+(\cos\alpha+\sec\alpha)^2=\tan^2\alpha+\cot^2\alpha+7$.

19. $\dfrac{1}{\sec A+\tan A}-\dfrac{1}{\cos A}=\dfrac{1}{\cos A}-\dfrac{1}{\sec A-\tan A}$

[Hint : Proceed as in solved Ex. 7.]

20. $\dfrac{\tan\alpha+\tan\beta}{\cot\alpha+\cot\beta}=\tan\alpha\cdot\tan\beta$.

21. $\sin^2 A\cos^2 B-\cos^2 A\sin^2 B=\sin^2 A-\sin^2 B$
$=\sin^2 A\sin^2 B(\operatorname{cosec}^2 B-\operatorname{cosec}^2 A)$

22. $\cot^2\theta.\dfrac{\sec\theta-1}{1+\sin\theta}+\sec^2\theta.\dfrac{\sin\theta-1}{1+\sec\theta}=0$.

23. $\sin^2 A.\tan A+\cos^2 A.\cot A+2\sin A.\cos A=\tan A+\cot A$

[Hint : Simplify L. H. S. and R. H. S. and prove that the results are equal.]

24. $\cot^4 A+\cot^2 A=\operatorname{cosec}^4 A-\operatorname{cosec}^2 A$.

25. $\dfrac{\cos^2\alpha-\sin^2\beta}{\sin^2\alpha\sin^2\beta}=(\cot^2\alpha-\tan^2\beta)\cot^2\beta$.

26. If $\cos\theta-\sin\theta=\sqrt{2}\sin\theta$, prove that
$\cos\theta+\sin\theta=\sqrt{2}\cos\theta$. (*P. U. 1941*)

Hint: $\cos\theta=(\sqrt{2}+1)\sin\theta \quad \therefore \sin\theta=\dfrac{1}{\sqrt{2}+1}\cos\theta$

$$=\frac{\sqrt{2}-1}{(\sqrt{2}-1)(\sqrt{2}+1)}\cos\theta \text{ (by rationalizing)}$$

2·3. Limits to the values of trigonometrical functions.

(a) To prove that $\sin\theta$ and $\cos\theta$ cannot be numerically greater than 1.

We know that for all values of θ

$$\sin^2\theta+\cos^2\theta=1.$$

Now $\sin^2\theta$ and $\cos^2\theta$ being both squares, are both necessarily positive. Hence since their sum is unity, neither the sine nor the cosine can be numerically greater than unity.

Cor. 1. The maximum value of $\sin\theta$ or $\cos\theta$ is 1 and the minimum value is -1.

$\because$ $\sin\theta$ cannot be numerically greater than 1 $\therefore$ it must lie between -1 and $+1$.

$\therefore$ the max. value of $\sin\theta$ is 1 and the minimum value is -1.

Similarly for $\cos\theta$.

Cor. 2. When $\sin\theta=\pm1$, $\cos\theta=0$.

$\cos\theta=\pm1$, $\sin\theta=0$.

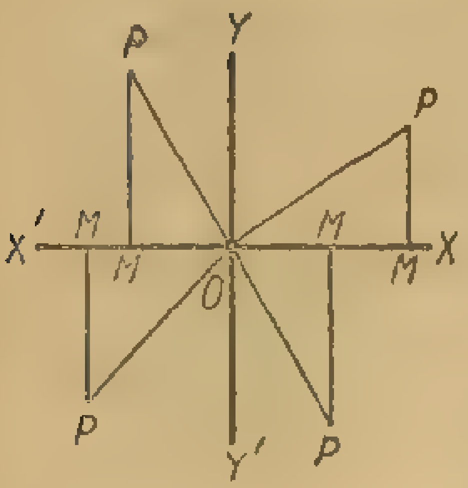
(b) To prove that $\sec\theta$ and $\operatorname{cosec}\theta$ cannot be numerically less than 1.

Since $\cos\theta$ cannot be greater than unity, therefore $\sec\theta$ which equals $\dfrac{1}{\cos\theta}$, cannot be numerically less than unity. So $\operatorname{cosec}\theta$ which equals $\dfrac{1}{\sin\theta}$, cannot be numerically less than unity.

(c) To prove that $\tan\theta$ and $\cot\theta$ can have any value.

$\because$ $\sec^2\theta=1+\tan^2\theta$ $\because$ $\tan^2\theta=\sec^2\theta-1$, but $\sec^2\theta$ cannot be less than 1 and can have any value greater than or equal to 1. $\therefore$ $\tan^2\theta$ cannot be negative and can have any positive value or value zero. Hence $\tan\theta$ can have any value $+$ive or $-$ive or zero.

Cot θ being reciprocal of $\tan\theta$, may similarly have any value.



Second Method. (Geometrical).

The foregoing results follow easily from the figure drawn. For, whatever be the value of the angle XOP neither the side OM nor the side MP is ever greater than OP.

Since MP is never greater than OP, the ratio $\frac{MP}{OP}$ is never greater than unity, so that the sine of an angle is never greater than unity.

Also since OM is never greater than OP, the ratio $\frac{MP}{OP}$ is never greater than unity, *i.e.*, the cosine of an angle is never greater than unity.

Again $\because$ OP is never less than OM numerically

$\therefore$ $\frac{OP}{OM}$ " " 1 " .

i.e., sec θ " " 1 "

Similarly cosec θ " " 1 "

As no restrictions can be put on the ratio $\frac{MP}{OM}$ or $\frac{OM}{MP}$

$\therefore$ tan θ and cot θ may have any value whatsoever.

Hence (*i*) sin θ and cos θ are both < 1

(*ii*) sec θ and cosec θ are both > 1

(*iii*) tan θ and cot θ can have any value.

EXERCISE II (B)

1. Are the following equations possible ?

(*i*) $\sin\theta = \frac{5}{4}$. (*ii*) $\cot\theta = \frac{1}{2}$. (*iii*) $\sec\theta = 3$.

Sol : (*i*) as sin θ is always numerically less than unity.

$\therefore$ it is impossible.

(*ii*) cot θ can have any value $\therefore$ it is possible.

(*iii*) sec θ is always greater than unity.

$\therefore$ it is possible.

2. Are the equations (*i*) $3\sin^2\theta + 5\sin\theta - 2 = 0$.

(*ii*) $2\cos^2\theta + \cos\theta - 6 = 0$ possible.

(*i*) **Sol.** $\sin\theta = \frac{-5 \pm \sqrt{25+24}}{6} = \frac{-5 \pm 7}{6} = \frac{1}{3}$ or -2.

$$\left[\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$$

One value of $\sin\theta=\frac{1}{2}$ is possible.

$\therefore$ the equation is possible.

3. Is the equation $2\sin\theta=a+\frac{1}{a}$ possible for all real values of a? (*P. U. 1939-S*)

[**Sol:** Writing the equation as a quadratic in 'a' we get $a^2-2a\sin\theta+1=0$ $\therefore$ a is real $\therefore$ the discriminant (b^2-4ac) is +ve.

$\therefore$ $4\sin^2\theta-4>0$ or $\sin^2\theta>1$, which is impossible. Hence the equation is not possible.]

4. Prove that the equation $\cos\theta=x+\frac{1}{x}$ is impossible, if x be real.

5. Is the statement $\operatorname{cosec}\theta=\frac{a^2+b^2}{2ab}$ possible? (*P. U. 1944-S*)

Sol: Now $\operatorname{cosec}\theta\geqslant 1$

$$\therefore \frac{a^2+b^2}{2ab}\geqslant 1$$

or $a^2+b^2\geqslant 2ab$ or $a^2+b^2-2ab\geqslant 0$

or $(a-b)^2\geqslant 0$, which is always true.

$\because$ a perfect square is alway +ve and hence $\geqslant 0$.

6. Is the statement $\sec\theta=\frac{x^2+y^2}{2xy}$ possible?

7. Show that $\sec^2\theta=\frac{4xy}{(x+y)^2}$ is only possible if $x=y$. (*Patna U. 1938-S*)

Sol: $\sec^2\theta\geqslant 1$

$\therefore$ $\frac{4xy}{(x+y)^2}\geqslant 1$ or $4xy\geqslant(x+y)^2$

or $0\geqslant(x+y^2)-4xy$

$\geqslant(x-y)^2$

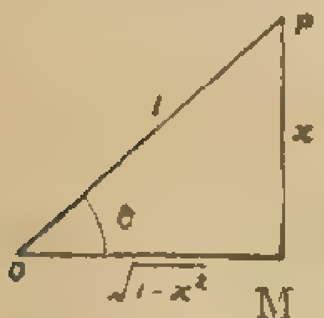
There are two possibilities (*i*) either $0>(x-y)^2$ or (*ii*) $0=(x-y)^2$.

(*i*) Now $0\not>(x-y^2)$ $\because$ $(x-y^2)$ is a perfect square which is always +ve

(*ii*) $\therefore$ $0=(x-y)^2$ or $x=y$.

2·4. Expression of any Trigonometric Ratios in terms of any other.

Example 1. *Express all the trigonometrical ratios in terms of the sine.*



First Method. (By the use of a triangle)
Let OMP be any rt-∠d △ and let ∠MOP $=\theta$ and let $\sin\theta = x = \dfrac{x}{1}$ ∴ the length OP is unity and the corresponding length of PM is x. Find the third side in terms of x.

$$\therefore\quad OM = \sqrt{OP^2 - MP^2} = \sqrt{1-x^2}$$

Hence $\sin\theta = \dfrac{MP}{OP} = \dfrac{x}{1} = \sin\theta.$

$$\cos\theta = \frac{OM}{OP} = \pm\frac{\sqrt{1-x^2}}{1} = \pm\sqrt{1-x^2} = \pm\sqrt{1-\sin^2\theta}$$

$$\tan\theta = \frac{MP}{OM} = \pm\frac{x}{\sqrt{1-x^2}} = \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

$$\cot\theta = \frac{OM}{MP} = \pm\frac{\sqrt{1-x^2}}{x} = \pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$$

$$\sec\theta = \frac{OP}{OM} = \pm\frac{1}{\sqrt{1-x^2}} = \pm\frac{1}{\sqrt{1-\sin^2\theta}}$$

$$\operatorname{cosec}\theta = \frac{OP}{MP} = \frac{1}{x} = \frac{1}{\sin\theta}$$

The last five equations give what is required.

Second Method. (By use of formulae).
Since $\sin^2\theta + \cos^2\theta = 1$,

$$\therefore\ \cos^2\theta = 1 - \sin^2\theta$$

or $\cos\theta = \pm\sqrt{1-\sin^2\theta}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}};\quad \cot\theta = \pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} = \pm\frac{1}{\sqrt{1-\sin^2\theta}};\quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

Note 1. *The first method can be employed as follows for all t-ratios.*

Let OMP be a right-angled $\triangle$ and $\angle MOP=\theta$. Let the given t=ratio$=x$ and put down its value in terms of the sides of the triangle OMP.

Let the denominator$=1$, then the numerator$=x$. Then by Pythagoras' Theorem, find the third side of the $\triangle$ OMP, taking the $\pm$ sign with the square root.

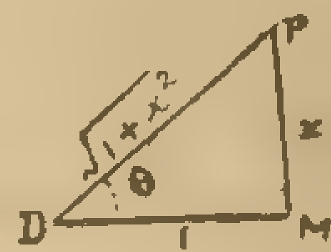
Now put down the value of the other ratios in terms of the sides of the triangle and replace x by the ratio in terms of which the other ratios are to be expressed.

Note 2. The figure has been drawn only for the case when θ is acute. The same method applies for angles of any magnitude.

Example 2. *Express all the trigonometrical ratios of θ in terms of the tangent θ.*

Let $\tan\theta=x=\dfrac{x}{1}$

$\therefore$ Put perp.$=x$ and base$=1$



Taking the usual figure of rt.-$\angle d$ $\triangle$ OMP, length OM is unity and let corresponding value of MP be x.

Then hyp. $OP=\pm\sqrt{OM^2+MP^2}$

$$=\pm\sqrt{1+x^2}$$

$$\text{Then } \sin\theta=\frac{MP}{OP}=\pm\frac{x}{\sqrt{1+x^2}}=\pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$\cos\theta=\frac{OM}{OP}=\pm\frac{1}{\sqrt{1+x^2}}=\pm\frac{1}{\sqrt{1+\tan^2\theta}}$$

$$\cot\theta=\frac{OM}{OP}=\frac{1}{x}=\frac{1}{\tan\theta}$$

$$\sec\theta=\frac{OP}{OM}=\pm\frac{\sqrt{1+x^2}}{1}=\pm\sqrt{1+\tan^2\theta}$$

$$\operatorname{cosec}\theta=\frac{OP}{MP}=\pm\frac{\sqrt{1+x^2}}{x}=\pm\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$$

Alternative Method $\therefore\ \sec^2\theta=1+\tan^2\theta$

$$\therefore\ \sec\theta=\pm\sqrt{1+\tan^2\theta}$$

$$\therefore \quad \cos\theta = \pm\frac{1}{\sqrt{1+\tan^2\theta}}, \quad \cot\theta = \frac{1}{\tan\theta}$$

$$\sin^2\theta = 1-\cos^2\theta = 1-\frac{1}{1+\tan^2\theta}$$

$$= \frac{1+\tan^2\theta-1}{1+\tan\theta} = \frac{\tan^2\theta}{1+\tan^2\theta}$$

$$\therefore \quad \sin\theta = \pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$\text{and } \operatorname{cosec}\theta = \pm\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$$

Example 3. *Express the circular functions of θ in terms of $\sec\theta$.*

Let $\sec\theta = x = \frac{x}{1}$ $\therefore$ Put hyp. $= x$ and base $= 1$

Then in the $\triangle$ OMP, $OP = x$ and $OM = 1$

$$\therefore \quad MP = \pm\sqrt{OP^2 - OM^2}$$

$$= \pm\sqrt{x^2-1}$$

$$\therefore \quad \sin\theta = \frac{MP}{OP} = \pm\frac{\sqrt{x^2-1}}{x}$$

$$= \pm\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$$

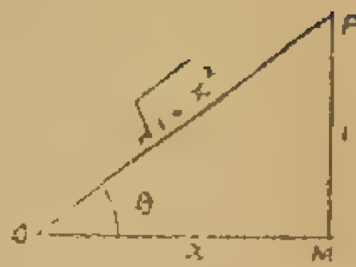
$$\cos\theta = \frac{OM}{OP} = \frac{1}{x} = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{MP}{OM} = \pm\frac{\sqrt{x^2-1}}{1} = \pm\sqrt{\sec^2\theta-1}$$

$$\cot\theta = \frac{OM}{MP} = \pm\frac{1}{\sqrt{x^2-1}} = \pm\frac{1}{\sqrt{\sec^2\theta-1}}$$

$$\operatorname{cosec}\theta = \frac{OP}{MP} = \pm\frac{x}{\sqrt{x^2-1}} = \pm\frac{\sec\theta}{\sqrt{\sec^4\theta-1}}$$

Example 4. *Express all the trigonometrical ratios in terms of cotangent θ.*



Let $\cot\theta = x = \frac{x}{1}$

$\therefore$ Put base $= x$ and perp. $= 1$

In the $\triangle$ OMP, OM $= x$, MP $= 1$

$\therefore \quad OP = \pm\sqrt{OM^2+MP^2} = \pm\sqrt{x^2+1}$

$$\therefore \sin\theta = \frac{MP}{OP} = \pm\frac{x}{\sqrt{x^2+1}} = \pm\frac{1}{\sqrt{\cot^2\theta+1}}$$

$$\cos\theta = \frac{OM}{OP} = \pm\frac{x}{\sqrt{x^2+1}} = \pm\frac{\cot\theta}{\sqrt{\cot^2\theta+1}}$$

$$\tan\theta = \frac{MP}{OM} = \frac{1}{x} = \frac{1}{\cot\theta}$$

$$\sec\theta = \frac{OP}{OM} = \pm\frac{\sqrt{x^2+1}}{x} = \pm\frac{\sqrt{\cot^2\theta+1}}{\cot\theta}$$

$$\operatorname{cosec}\theta = \frac{OP}{MP} = \pm\frac{\sqrt{x^2+1}}{1} = \pm\sqrt{\cot^2\theta+1}$$

Second method.

$\because \quad \operatorname{cosec}^2\theta = 1 + \cot^2\theta$

$\therefore \quad \operatorname{cosec}\theta = \pm\sqrt{1+\cot^2\theta}$

and $\sin\theta = \pm\dfrac{1}{\sqrt{1+\cot^2\theta}}$

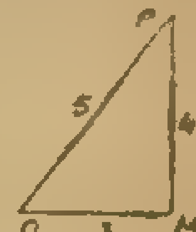
$$\cos\theta = \cot\theta\sin\theta$$

$$= \pm\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$$

$$\sec\theta = \frac{1}{\cos\theta} = \pm\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$$

$$\tan\theta = \frac{1}{\cot\theta}.$$

Example 5. *If* $\cos\theta = \frac{3}{5}$, *find the values of the other ratios.*



Let OMP be a right angled $\triangle$ and $\angle$MOP $= \theta$.

Then $\cos\theta = \dfrac{OM}{OP} = \dfrac{3}{5}$

(given)

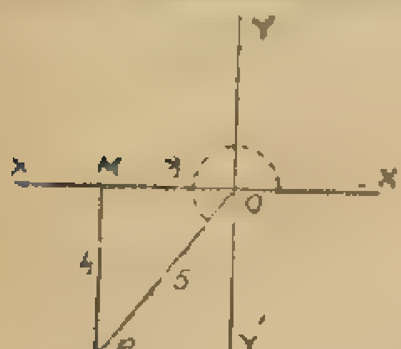
Let the denominator OP $= 5$

then OM $= 3$

$$\therefore \quad MP = \pm\sqrt{OP^2 - OM^2}$$
$$= \pm\sqrt{25-9} = \pm 4$$

$$\therefore \quad \sin\theta = \frac{MP}{OP} = \pm\frac{4}{5}$$
$$\tan\theta = \frac{MP}{OM} = \pm\frac{4}{3}$$
$$\cot\theta = \frac{OM}{MP} = \pm\frac{3}{4}$$
$$\sec\theta = \frac{OP}{OM} = \frac{5}{3}$$
$$\operatorname{cosec}\theta = \frac{OP}{MP} = \pm\frac{5}{4}.$$

Example 6. *Given that* $\tan\theta = \frac{4}{3}$ *where* θ *lies in the third quadrant, find the other trigonometrical ratios of* θ.



Here $\tan\theta = \frac{4}{3}$ and also $\tan\theta = \frac{MP}{OM}$.

Let $MP = 4$ in magnitude so that $OM = 3$ in magnitude.

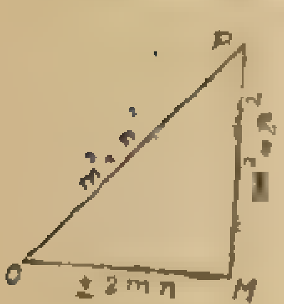
Hence $OP = \sqrt{16+9} = 5$.

Now since the angle is in the third quadrant, therefore its sine, cosine, secant and cosecant are negative while its tangent and cotangent are positsve or from the figure OM is $-$ve and $\therefore = -3$ and MP is also negative $\therefore = -4$.

$$\text{Hence } \sin\theta = \frac{MP}{OP} = -\frac{4}{5}$$
$$\cos\theta = \frac{OM}{OP} = -\frac{3}{5}$$

$\operatorname{cosec}\theta = -\frac{5}{4}$; $\sec\theta = -\frac{5}{3}$, $\cot\theta = \frac{3}{4}$.

Example 7. *If* $\sin\theta = \frac{m^2-n^2}{m^2+n^2}$, *find all the other trigonometrical ratios in terms of m and n.* (D.U. 1941)



In the right-angled triangle OMP, let $\angle MOP = \theta$.

$$\text{Then } \sin\theta = \frac{MP}{OP};$$
$$\text{Also } \sin\theta = \frac{m^2-n^2}{m^2+n^2}$$

Let the denominator $OP = m^2 + n^2$

then $MP = m^2 - n^2$

$$\therefore \quad OM = \pm\sqrt{OP^2 - MP^2}$$
$$= \pm\sqrt{(m^2+n^2)^2 - (m^2-n^2)^2}$$
$$= \pm\sqrt{m^4+n^4+2m^2n^2-m^4-n^4+2m^2n^2}$$
$$= \pm\sqrt{4m^2n^2} = \pm 2mn.$$

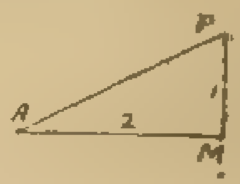
$$\therefore \cos\theta = \frac{OM}{OP} = \pm\frac{2mn}{m^2+n^2}; \ \tan\theta = \pm\frac{m^2-n^2}{2mn}$$

$$\operatorname{cosec}\theta = \frac{m^2+n^2}{m^2-n^2}.$$

Example 8. *If* $\cos A = 2\sin A$, *find* $\operatorname{cosec} A$.

(*P. U. 1943*)

$\text{Cos } A = 2\sin A$



Dividing by $\cos A$, we get $1 = 2\dfrac{\sin A}{\cos A}$

$= 2\tan A$

$\therefore \tan A = \frac{1}{2}$

In the right-angled $\triangle$ MAP, let $PM = 1$ and $AM = 2$ then $\tan\angle PAM = \frac{1}{2}$ and $AP = \pm\sqrt{1+4} = \pm\sqrt{5}$

$$\therefore \operatorname{cosec} A = \frac{AP}{MP} = \pm\frac{\sqrt{5}}{1} = \pm\sqrt{5}.$$

Second Method. $\text{Cos } A = 2\sin A$. Dividing by $\sin A$, we get

$\text{Cot } A = 2$. Now $\operatorname{cosec}^2 A = 1 + \cot^2 A$

$= 1 + 4 = 5$

$\therefore \operatorname{cosec} A = \pm\sqrt{5}$

EXERCISES II (c)

1. Express all the other trigonometrical ratios in terms of the cosine. (*P. U.*)

2. If $6\cos^2\theta = 1$, find the other trigonometric ratios. (*P. U. 1947*)

[**Hint :** $\cos\theta = \pm\dfrac{1}{\sqrt{6}}$, by taking the square root].

3. If $5\sin^2\theta - 1 = 0$, find the other trigonometric ratios. (*P. U. 1946*)

4. Find the other trigonometric ratios if

(*i*) $\tan\theta = \frac{1}{\sqrt{3}}$. (*P. U. 1945*)

(*ii*) $\tan\theta = \frac{2}{3}$. (*P. U. 1947-S*)

5. If $\sec A = \frac{41}{9}$, find the other *t*-ratios (*C. U.*)

6. If $\cos A = \frac{m^2 - n^2}{m^2 + n^2}$, find sin A and tan A (*P. U.*)

7. Given $\tan\theta = \frac{2ab}{a^2 - b^2}$, find the other circular functions of θ. (*P. U.*)

8. If $\frac{1-\sin A}{1+\sin A} = \frac{a^2}{b^2}$, find tan A.

[**Hint :** Find sin A by Comp. and Dividendo].

9. Find the sine and secant of an angle between 270° and 360° whose cotangent is equal to $-\frac{a}{b}$, a and b being positive quantities.

10. If $\sin\theta = \frac{21}{29}$, prove that $\sec\theta + \tan\theta = 2\frac{1}{2}$, θ lying between 0° and 90°. What will be the value of the expression when θ lies between (*i*) 90° and 180°, (*ii*) and lies between 180° and 270° ? (*D. U. 1937*)

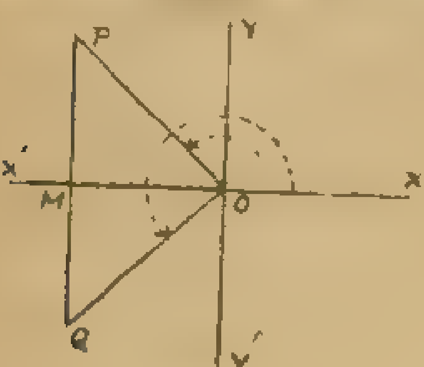
[**Hint :** (*i*) θ lies in the second quadrant $\therefore$ the values of $\tan\theta$ and $\sec\theta$ are both negative etc.

(*ii*) $\sin\theta$ is given to be +ve, hence it does not lie in the third quadrant. Hence there is no value for this part.]

11. An angle lies between 180° and 270° and $\tan\alpha = \frac{24}{7}$, find the other trigonometric ratios. (*P. U.*)

[**Hint :** Proceed as in solved Ex. 6]

12. If $\cos A = \frac{\sqrt{3}}{2}$, find tan A, drawing a diagram to explain the two values. (*P. U. 1933-4S*)



[**Hint :** cos A is −ve $\therefore$ A lies in the second or third quadrant as ∠XOP or ∠XOQ.

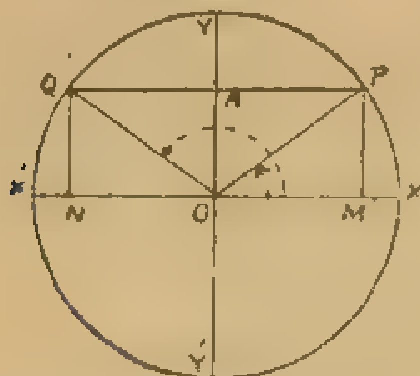
$\therefore$ $OM = -\sqrt{3}$, $OP = 2$

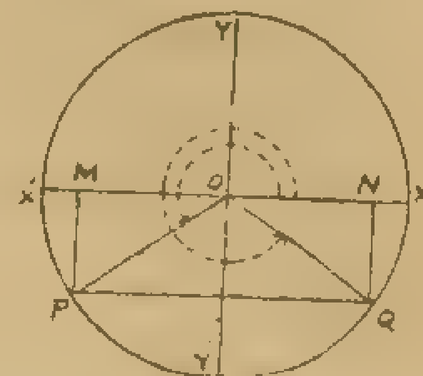
$\therefore$ $PM = \sqrt{4-3} = +1$

and $MQ = -1$.]

2·5. Construction of an angle having a given t-ratio.

2·51. Given the sine or cosecant of an angle, to construct the angle.





Let X'OX and Y'OY be two ⊥ lines respectively. Let the sine of the angle = S where S is a positive or negative quantity less than 1 numerically.

With O as centre and radius equal to unity describe a circle and cut off the length OA equal to S in magnitude from the positive or negative side of the OY according as S is + or −. Through the point A draw a line QAP parallel to X'OX meeting the circle in the points Q and P. Draw PM and QN perpendiculars to X'OX and join OP and OQ. Then ∠XOP and ∠XOQ are angles having the given sine.

Proof.

sin ∠XOP $= \frac{MP}{OP} = \frac{S}{1} = S$ and sin XOQ $= \frac{NQ}{OQ} = \frac{S}{1} = S$. Either of the angles is a solution of the problem.

Note 1. The construction fails if S > 1 numerically.

Note 2. Between 0° and 360° there are two angles whose sine is equal to the given quantity S.

Note 3. If the cosecant be given, the sine is known and a similar construction will hold.

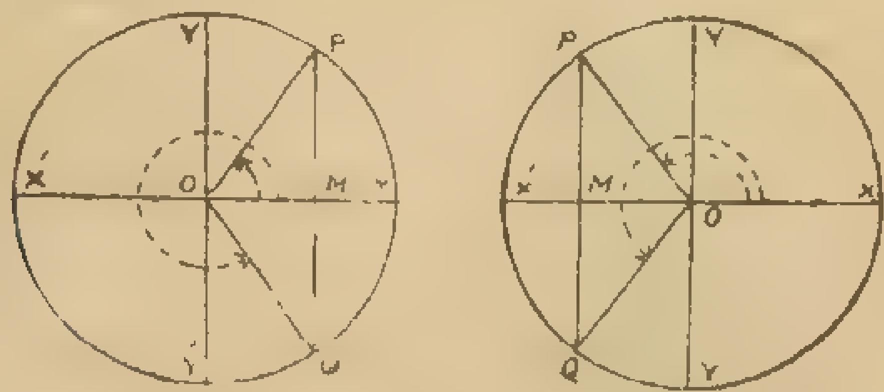
Note 4. The angles XOP and XOQ are equal if S = 1 or − 1.

Notation. An angle whose sine = S is denoted by $\sin^{-1}S$ [read as inverse sine S]. Similarly angle whose cosine is k is denoted by $\cos^{-1}k$ and an angle whose tangent is t is denoted by $\tan^{-1}t$ and so on.

$\sin^{-1}k$ and $\tan^{-1}t$ are called inverse circular functions,

Note it carefully that $\sin^{-1}S$ is not equal to $(\sin S)^{-1}$ or $\frac{1}{\sin S}$.

2.52. Given the cosine or secant of angle; to construct the angle.



Let the cosine = C, C being + or − but always less than 1 numerically.

Let X′OX and Y′OY be the two ⊥ lines respectively.

With O as centre and unity as radius, describe a circle, and cut off the length OM equal to C from OX or OX′ according as C is + or −. Through the point M draw PMQ perpendicular to X-axis meeting the circle in P and Q. Join OP and OQ. Then ∠XOP and ∠XOQ are angles having the given cosine.

Proof. $\text{Cos XOP} = \frac{OM}{OP} = C$ and $\cos XOQ = \frac{OM}{OQ} = C$. Either angle will be a solution of the problem.

Note 1. The construction fails if $C > 1$ numerically.

Note 2. If the secant be given, the cosine is its reciprocal and the construction is the same as before.

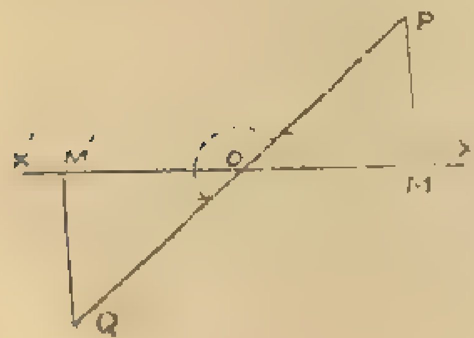
Note 3. There are two angles lying between 0° and 360° which have a given cosine or secant (*viz*., angles XOP and XOQ)

Note 4. If $C = +1$ or -1, then angles XOP and XOQ are equal.

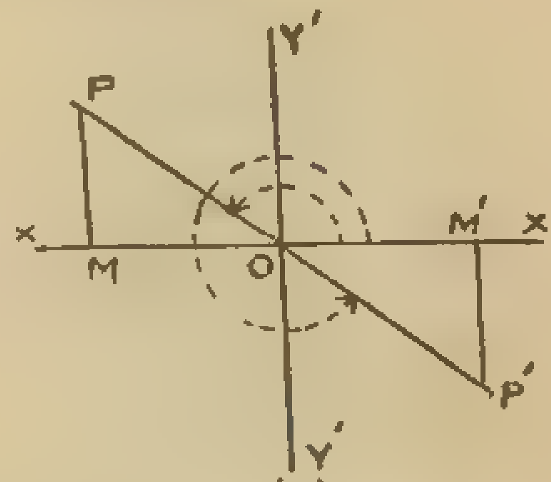
2·53. Given the tangent or cotangent of an angle; to construct the angle.

Let tangent of the angle be equal to t where t may be any positive or negative number.

Draw a straight line X'OX, cut off OM=1.



(*i*)



(*ii*)

Case (*i*) If *t* is +, draw MP upwards ⊥ X'OX and cut it off equal to *t* as in figure (*i*).

Case (*ii*) If *t* is −, draw MP downwards ⊥ X'OX and cut it off equal to *t* as in Figure (*ii*).

Produce PO to Q. Then ∠XOP or ∠XOQ is the required angle.

Proof. Cut off OQ=OP and draw QM' ⊥ X'OX.

Then $\tan XOP=\dfrac{MP}{OM}=\dfrac{t}{1}=t$

and $\tan XOQ=\dfrac{M'Q}{OM'}=\dfrac{-MP}{-OM}=t.$ [∵ △s OMP and OQM' are congruent.]

Note 1. In this case the construction never fails.

Note 2. If the co-tangent of an angle be given, its tangent is known and a similar construction holds in that case also.

Note 3. There are two angles lying between 0° and 360° which have a given tangent or co-tangent. (*Viz*., angle XOP and XOQ).

Example 1. *At a point O construct two consecutive (acute) angles whose tangents are* $\frac{3}{4}$ *and* $\frac{2}{5}$. (*P.U. 1935*)

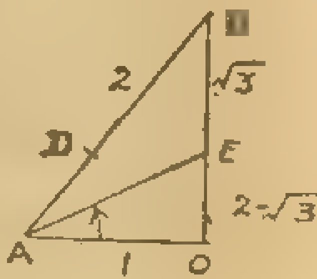
(*i*) Draw a straight line X'OX. On OX cut off OM=4. At M draw MP ⊥ X'OX and cut it off equal to 3. Join OP. Then ∠XOP is the first required angle.

(*ii*) Draw PQ ⊥ OP and =2. Join OR. The ∠POQ is the second required angle.

EXERCISE II (*d*)

1. Construct an acute angle whose cosine in $\frac{2}{3}$ and find its tangent and cosecant.

2. Construct an angle whose tangent is $2-\sqrt{3}$.



Sol. Take a rt. $\angle$d $\triangle$ where base$=1$ unit;

and hyp. $AB=2$ units; then the $\perp$ is $\sqrt{3}$

BA cut off $BD=BO=\sqrt{3}$;

then $AD=2-\sqrt{3}$ units.

From OB cut off $OE=AD$

$\angle OAE$ is the reqd. $\angle$.

MISCELLANEOUS EXERCISES ON CHAPTER II

1. Prove that $\dfrac{1+\tan^2\theta}{1+\cot^2\theta}=\dfrac{(1-\tan\theta)^2}{(1-\cot\theta)^2}$. (*P.U.*)

[**Hint:** Change $\cot\theta$ into $\tan\theta$ and prove that R.H.S. $=\tan^2\theta=$ L.H.S.]

2. Prove that

$$\sec^6\theta-\tan^6\theta=1+3\tan^2\theta\sec^2\theta \qquad (P.U.\ 1950)$$

[**Hint:** Factorise the L.H.S.]

3. Show that

$$\sin^6\theta+\sin^4\theta\cos^2\theta-\sin^2\theta\cos^4\theta-\cos^6=\sin^2\theta-\cos^2\theta$$

(*D.U. 1941*)

4. Prove that

$$(1-\tan\theta)^2+(1-\cot\theta)^2=(\sec\theta-\operatorname{cosec}\theta)^2. \qquad (P.U.)$$

5. If $\tan A+\sin A=m$

and $\tan A-\sin A=n$, show that

$$m^2-n^2=4\sqrt{mn} \qquad (P.U.\ 1939\text{-}S)$$

[**Sol.** L. H. S.$=m^2-n^2$

$$=(\tan A+\sin A)^2-(\tan A-\sin A)^2$$
$$=4\tan A\sin A.$$

$$\text{R. H. S.}=4\sqrt{(\tan A+\sin A)(\tan A-\sin A)}$$
$$=4\sqrt{\tan^2 A-\sin^2 A}$$
$$=4\sqrt{\frac{\sin^2 A}{\cos^2 A}-\sin^2 A}=4\sin A\sqrt{\frac{1}{\cos^2 A}-1}$$
$$=4\sin A\sqrt{\sec^2 A-1}=4\sin A\tan A.$$

6. If $\sin A = \frac{2ab}{a^2+b^2}$ and $\sin B = \frac{2cd}{c^2+d^2}$,

prove that $\sin A \cos B + \cos A \sin B$

$$= \frac{2mn}{m^2+n^2}, \text{ when } m = ad+bc \text{ and } n = ac-bd.$$

7. If $\tan A + \sec A = 4$, find $\sin A$ and $\cos A$. (*P.U.*)

[**Hint :** $\sec^2 A - \tan^2 A = 1$. Divide it by $\tan A + \sec A = 4$; we get $\sec A - \tan A = \frac{1}{4}$. Find the value of $\tan A$ and $\sec A$, and hence of $\sin A$, by combining the given relation and the new relation].

8. Is the equation $\cos^2\theta = \frac{(a+b)^2}{4ab}$ possible? If so, when?

9. Prove that $\sec^2\theta + \cos^2\theta$ can never be less than 2. (*P. U. 1940*)

[**Sol.** $\sec^2\theta + \cos^2\theta = \sec^2\theta + \cos^2\theta - 2 + 2$

$$= \sec^2\theta + 2\cos^2\theta - 2(\sec\theta \times \cos\theta) + 2$$

$$= (\sec\theta - \cos\theta)^2 + 2.$$

$\therefore$ $\sec^2\theta + \cos^2\theta$ is always > 2, because $(\sec\theta - \cos\theta)^2$ is always $+$ve; it $= 2$, when $\sec\theta = \cos\theta$.

Hence $\sec^2\theta + \cos^2\theta \not< 2$.]

10. If $\sin x + \sin^2 x = 1$, find $\sin x$. Also prove that

$$\cos^2 x + \cos^4 x = 1 \qquad (A. U.)$$

[**Hint :** $\sin x = 1 - \sin^2 x = \cos^2 x$. Square and put $\sin^2 x = 1 - \cos^2 x$].

11. Eliminate θ from the following equations

(*i*) $x = a\cos\theta$, $y = b\sin\theta$ (*ii*) $x = a\sec\theta$, $y = a\tan\theta$

Bring the *t*-ratio on one side; then.

[**Hint :** (*i*) Square and add

(*ii*) Divide by a the 1st equation and by b the 2nd equation and then square and subtract].

12. If $\sec\theta = x + \frac{1}{4x}$, prove that $\sec\theta + \tan\theta = 2x$ or $\frac{1}{2x}$.

[**Hint :** Find $\tan\theta$ from the given relation :

$\therefore$ $\tan\theta = \pm\frac{(4x^2-1)}{4x}$ or $= \pm\left(x - \frac{1}{4x}\right)$].

13. If $\sin\theta = \frac{5}{13}$, find $\tan\theta + \sec\theta$. (*P.U.*)

14. If $5 \tan \theta = 4$, find the value of $\dfrac{5 \sin \theta - 3 \cos \theta}{\sin \theta + 2 \cos \theta}$
(*P. U.*)

[Hint : Divide the numerator and denominator by $\cos \theta$].

15. If $\sin \theta + \cos \theta = p$ and $\sin \theta - \cos \theta = q$, prove that
$$p^2 + q^2 = 2.$$

ANSWERS TO QUESTIONS IN CHAPTER II

Exercises Art, 8'2

1. (*i*) 1st and 2nd. (*ii*) second and third.
(*iii*) 2nd and fourth. (*iv*) 1st and 4th.
(*v*) 3rd and 4th.

2. (*i*) All +ve.
(*ii*) sin, cosine, sec and cosec −ve tan and cot +ve
(*iii*) sin and cosecant +ve, others are −ve.

Exercises II (*b*)

6. Possible.

Exercises II (*c*)

1. $\sin \theta = \pm\sqrt{1-\cos^2\theta}$; $\tan \theta = \pm\dfrac{\sqrt{1-\cos^2\theta}}{\cos \theta}$

$\cot \theta = \pm\dfrac{\cos \theta}{\sqrt{1-\cos^2\theta}}$; $\sec \theta = \dfrac{1}{\cos \theta}$; $\operatorname{cosec} \theta = \pm\dfrac{1}{\sqrt{1-\cos^2\theta}}$

2. $\sin \theta = \pm\sqrt{\frac{5}{6}}$ $\tan \theta = \pm\dfrac{\sqrt{5}}{1}$, $\cot \theta = \pm\dfrac{1}{\sqrt{5}}$;

$\sec \theta = \pm\dfrac{\sqrt{6}}{1}$, $\operatorname{cosec} \theta = \pm\dfrac{\sqrt{6}}{\sqrt{5}}$.

3. $\operatorname{cosec} \theta = \pm\dfrac{2}{\sqrt{5}}$; $\tan \theta = \pm\frac{1}{2}$; $\cot \theta = \pm 2$; $\sec \theta = \pm\dfrac{\sqrt{5}}{2}$

$\operatorname{cosec} \theta = +\sqrt{5}$.

4. (*i*) $\sin \theta = \pm\frac{1}{2}$; $\cos \theta = \pm\dfrac{\sqrt{3}}{2}$; $\cot \theta = \sqrt{3}$;

$\sec \theta = \pm\dfrac{2}{\sqrt{3}}$, $\operatorname{cosec} \theta = \pm 2$.

(*ii*) $\sin \theta = \pm\dfrac{2}{\sqrt{13}}$; $\cos \theta = \pm\dfrac{3}{\sqrt{13}}$ $\cot \theta = \frac{3}{2}$;

$\sec \theta = \pm\dfrac{\sqrt{13}}{3}$, $\operatorname{cosec} \theta = \pm\dfrac{\sqrt{13}}{2}$

5. $\sin\theta=\pm\frac{40}{41}$; $\cos\theta=\frac{9}{41}$; $\tan\theta=\pm\frac{40}{9}$; $\cot\theta=\pm\frac{9}{40}$; $\operatorname{cosec}\theta=\pm\frac{41}{40}$.

6. $\sin A=\pm\dfrac{2mn}{m^2+n^2}$; $\tan A=\pm\dfrac{2mn}{m^2-n^2}$.

7. $\sin\theta=\pm\dfrac{2ab}{a^2+b^2}$; $\cos\theta=\pm\dfrac{a^2-b^2}{a^2+b^2}$; $\cot\theta=\dfrac{a^2-b^2}{2ab}$;

$\sec\theta=\pm\dfrac{a^2+b^2}{a^2-b^2}$; $\operatorname{cosec}\theta=\pm\dfrac{a^2+b^2}{2ab}$.

8 $\tan A=\pm\dfrac{b^2-a^2}{2ab}$.

9. $-\dfrac{b}{\sqrt{a^2+b^2}}$; $\dfrac{\sqrt{a^2+b^2}}{a}$

11. $\sin\alpha=-\dfrac{24}{25}$; $\cos\alpha=-\dfrac{7}{25}$; $\cot\alpha=\dfrac{7}{24}$; $\sec\alpha=\dfrac{25}{7}$;

$\operatorname{cosec}\alpha=-\dfrac{25}{24}$.

12. $\tan A=\pm\dfrac{1}{\sqrt{3}}$.

Exercise II (*d*)

1. $\frac{4}{3}$; $\frac{6}{4}$.

Miscellaneous Examples on Chapter II

7. $\sin\theta=\frac{15}{17}$, $\cos\theta=\frac{8}{17}$.

8. Possible only if $a=b$.

10. $\sin x=\dfrac{-1\pm\sqrt{5}}{2}$; but as $\dfrac{-1-\sqrt{5}}{2}$ is numerically >1

$\therefore$ only one value $\sin x=\dfrac{-1+\sqrt{5}}{2}$ is permissible.

11. (*i*) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$; $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.

13. $\pm\frac{1}{3}$.

14. $\frac{7}{14}$.

CHAPTER III

TRIGONOMETRICAL RATIOS OF SOME STANDARD ANGLES, SIMPLE CASES OF HEIGHTS AND DISTANCES

3·1. To find the trigonometric ratios of 45° or $\frac{\pi}{4}$ radians.

Let $\angle XOP = 45°$. From P, any point in OP, draw $PM \perp OM$. Then $\angle OPM = 90° - 45 = 45°$

$\therefore$ OPM is an isos. $\triangle$

$\therefore OM = MP = a$ (say)

$\therefore OP = \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}$

$$\therefore \sin 45 = \frac{MP}{OP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$$\cos 45° = \frac{OM}{OP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$$\tan 45° = \frac{MP}{OM} = \frac{a}{a} = 1;$$

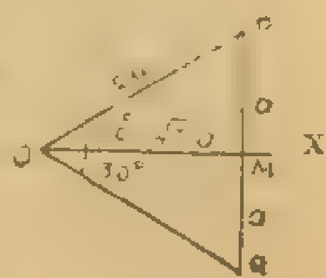
$$\cot 45° = \frac{OM}{MP} = \frac{a}{a} = 1;$$

$$\sec 45° = \frac{a\sqrt{2}}{a} = \sqrt{2};$$

$$\text{cosec } 45° = \frac{OP}{MP} = \frac{a\sqrt{2}}{2}\sqrt{2}.$$

3·2. To find the trigonometrical ratios of 30 or $\frac{\pi}{6}$ radians.

Let $\angle XOP$ be 30°. Make $\angle QOX = 30°$ in [illegible]. From any point P in OP draw $PM \perp OX$ and produce it to meet OQ in Q.



Then $\triangle$s MOP and MOQ are congruent.

[$\because \angle MOP = \angle MOQ$,

$\angle PMO = \angle QMO$ and OM is common.]

$\therefore$ $\angle P = \angle Q = 60°$, because $\angle OPM = 90° - \angle MOP$
$= 90° - 30° = 60$.

Hence $\triangle$ POQ is equilateral and side PM = side MQ

Let $MP = MQ = a$ (say)

$\therefore$ $OP = PQ = 2MP = 2a$

and from rt. $\angle$d $\triangle OPM$, $OM = \sqrt{OP^2 - MP^2} = \sqrt{4a^2 - a^2}$
$= a\sqrt{3}$.

$\therefore$ $\sin 30° = \dfrac{MP}{OP} = \dfrac{a}{2a} = \frac{1}{2}$; | $\cot 30° = \dfrac{OM}{MP} = \dfrac{a\sqrt{3}}{a} = \sqrt{3}$;

$\cos 30° = \dfrac{OM}{OP} = \dfrac{a\sqrt{3}}{2a} = \dfrac{\sqrt{3}}{2}$; $\sec 30° = \dfrac{OP}{OM} = \dfrac{2a}{a\sqrt{3}} = \dfrac{2}{\sqrt{3}}$

$\tan 30° = \dfrac{MP}{OM} = \dfrac{a}{a\sqrt{3}} = \dfrac{1}{\sqrt{3}}$; | $\operatorname{cosec} 30° = \dfrac{OP}{MP} = \dfrac{2a}{a} = 2$.

3·3 To find the trigonometrical ratios of 60° or $\dfrac{\pi}{3}$ radians.

Let $\angle POR = 60°$;

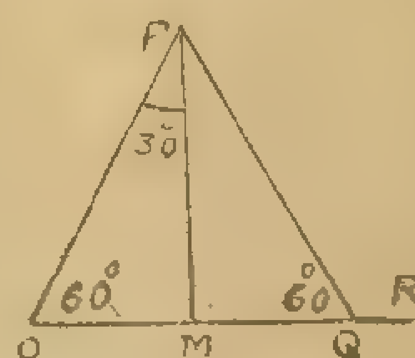
From P drop a $\perp$ PM to OR.

Cut off MQ from OR = OM.

Then $\triangle$s OPM and PMQ are $\equiv$

[$\because$ $OM = MQ$, $PM = PM$
and $\angle OMP = \angle PMQ$
= a rt. $\angle$].

$\therefore$ $\angle PQM = \angle MOP = 60°$



Hence OPQ is an equilateral $\triangle$

$\therefore$ $OP = OQ$ or $= 2\,OM$ [$\because$ $OM = MQ$]

Let $OM = a$ $\therefore$ $OP = 2a$

Hence from rt. $\angle$d. $\triangle$ OPM,

$PM = \sqrt{OP^2 - OM^2} = \sqrt{4a^2 - a^2} = \sqrt{3}a$

Hence $\sin 60 = \dfrac{MP}{OP} = \dfrac{a\sqrt{3}}{2a} = \dfrac{\sqrt{3}}{2}$; $\cot 60° = \dfrac{OM}{MP} = \dfrac{a}{a\sqrt{3}} = \dfrac{1}{\sqrt{3}}$;

$\cos 60° = \dfrac{OM}{OP} = \dfrac{a}{2a} = \frac{1}{2}$; $\sec 60° = \dfrac{OP}{OM} = \dfrac{2a}{a} = 2$;

(P. U. 1917-S.)

$\tan 60° = \dfrac{MP}{OM} = \dfrac{a\sqrt{3}}{a} = \sqrt{3}$; | $\operatorname{cosec} 60° = \dfrac{OP}{OM} = \dfrac{2a}{a\sqrt{3}}$
$= \dfrac{2}{\sqrt{3}}$.

Note. Since all the trigonometrical ratios of a positive acute angle are positive, therefore the signs of radicals in the values of the sides of the $\triangle$ OMP are all taken positive.

3·4. Trigonometric ratios of 0° and 90° and 180°. Before finding out the trigonometrical ratios of 0° and 90° it is necessary that the student should have the idea of infinity.

Meaning of $\frac{r}{0}$ or ∞.

In the ordinary sense $\frac{r}{0}$ is meaningless as division by zero is not allowed, but the following discussion will show that a meaning can be attached to it.

Let r be positive. Consider the series of quantities :

$$\frac{r}{1}, \frac{r}{0{\cdot}1}, \frac{r}{0{\cdot}01}, \frac{r}{0{\cdot}001}, \frac{r}{0{\cdot}0001}, \ldots\ldots$$

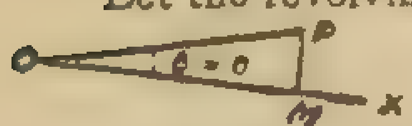
Their values are r, $10r$, $100r$, $1000r$, $10000r$,............... We notice that as the denominator of the fraction grows smaller and smaller, whilst the numerator remains the same, the value of the fraction grows larger and larger or in, other words, if the denominator decreases without limit, the value of the fraction increases without limit.

$\therefore$ when the denominator decreases and is ultimately nearly equal to 0, the value of the fraction increases in such a way that it exceeds any number, however large it may be, $\therefore$ we say that the value of the fraction is infinitely large and is denoted by ∞ (read 'infinity').

$\therefore \quad \frac{r}{0} = \infty$ similarly $\frac{-r}{0} = -\infty$.

3·5. To find the trigonometrical ratios of 0°.

Let the revolving line start from OX and trace out a very small angle XOP $=\theta$. From any point P in it draw PM $\perp$ OX. Then OPM is the triangle of reference. When the line OP moves back and coincides with OX, $\angle$XOP becomes equal to zero and MP $=0$.



Also OP $=$ OM $=a$ (say)

$\therefore \ \sin 0° = \dfrac{MP}{OP} = \dfrac{0}{OP} = \dfrac{0}{a} = 0 \cdot \qquad \cot 0° = \dfrac{OM}{MP} = \dfrac{OM}{0} = \dfrac{a}{0}$
$= \infty$;

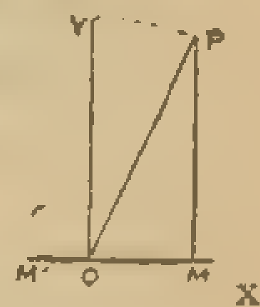
$\cos 0° = \dfrac{OM}{OP} = \dfrac{a}{a} = 1$; $\qquad \sec 0° = \dfrac{OP}{OM} = \dfrac{a}{a} = 1$;

$\tan 0° = \dfrac{MP}{OM} = \dfrac{0}{a} = 0$; $\qquad \operatorname{cosec} 0° = \dfrac{OP}{OM} = \dfrac{a}{0} = \infty$

Note These are in reality the limiting values of the *t* ratios of a positive acute angle when the angle tends to zero, because strictly speaking when the angle is 0°, there is no triangle of reference and there are no trigonometrical ratios.

3·6. To find the trigonometrical ratios of 90° or $\dfrac{\pi}{2}$ radians.

Let the revolving line OP start from OX and make an angle XOP which is very nearly equal to 90°. Then if PM be drawn ⊥ on OX, OPM is the triangle of reference.



And if XOP becomes exactly 90°, PM will coincide with OY or OP.

$\therefore \quad MP = OP = a$ (say)

and $OM = 0$

Hence $\sin 90° = \dfrac{MP}{OP} = \dfrac{a}{a} = 1$; $\cos 90° = \dfrac{OM}{OP} = \dfrac{0}{a} = 0$;

$$\tan 90° = \frac{MP}{+OM} = \frac{a}{+0} = +\infty ;$$ (*P. U. 1947-S*)

$$\cot 90° = \frac{OM}{MP} = \frac{0}{a} = 0$$

$$\sec 90° = \frac{OP}{OM} = \frac{a}{+0} = +\infty$$

$$\operatorname{cosec} 90° = \frac{OP}{MP} = \frac{a}{a}.$$

Note. These are really the limiting values of the trigonometrical ratios of a positive acute angle when the angle tends to 90°.

3·7. To find the trigonometrical ratios of 180° or π radians.

Let $\angle XOP$ be nearly equal to 180°. Then if PM be drawn $\perp$ XO produced, the ordinate MP will be small and the abscissa OM be nearly equal in length to OP; but the former is negative while the length OP is always +ve. Hence when $\angle XOP$ actually becomes equal to 180°, we have $MP=0$, $OM=-OP=-a$ where $OP=a$ (say)

$$\therefore \quad \sin 180° = \frac{MP}{OP} = \frac{0}{a} = 0;$$

$$\cos 180° = \frac{OM}{OP} = \frac{-a}{a} = -1; \quad \tan 180° = \frac{MP}{OM} = \frac{0}{OM} = 0$$

$$\cot 180° = \mp\infty$$

$$\operatorname{cosec} 180° = \pm\infty, \ \sec 180° = -1.$$

Note. (*i*) $\cot 180° = -\infty$ when the angle is slightly less than 180° as it then lies in the second quadrant.

$\cot 180° = +\infty$ when the angle is slightly greater than 180° as it then lies in the 3rd quadrant. Similar case is of cosec 180°.

(*ii*) These are in fact the limiting values of the *t*-ratios, for if $\angle = 180°$, no $\triangle$ is possible.

Note. In order to commit to memory the results, it should be observed that the values of all trigonometric ratios can be derived from the values of sine and cosine of that angle, for

$$\tan\theta = \frac{\sin\theta}{\cos\theta}; \ \cot\theta = \frac{\cos\theta}{\sin\theta}; \ \sec\theta = \frac{1}{\cos\theta}; \ \operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

But to facilitate the work of memorising, the important angles and their ratios are given in a tabular form.

Angle	0°	30°	45°	60°	90°
Sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Co-sine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Tangent	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Cotangent	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
Secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
Co-secant	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

It may be noticed with advantage that for the angles

$$0^\circ, 30^\circ, 45^\circ, 60^\circ, 90^\circ$$

(*i*) the sines are the square roots of

$$\tfrac{0}{4}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{4}{4}$$

(*ii*) the cosines are the square roots of

$$\tfrac{4}{4}, \tfrac{3}{4}, \tfrac{2}{4}, \tfrac{1}{4}, \tfrac{0}{4}.$$

ILLUSTRATIVE EXAMPLES

1. *Simplify* $\sin 30^\circ + \cos 60^\circ - \tan 45^\circ$.

We know that $\sin 30^\circ = \frac{1}{2}$; $\cos 60^\circ = \frac{1}{2}$ and $\tan 45^\circ = 1$.

$\therefore$ the given expression $= \frac{1}{2} + \frac{1}{2} - 1 = 0$.

2. *Simplify*

$$4 \sin^2 \frac{\pi}{4} + 4 \cos \frac{\pi}{3} - \operatorname{cosec}^2 \frac{\pi}{3} \cot^2 \frac{\pi}{6}$$

The given expression $= 4 \left(\frac{1}{\sqrt{2}}\right) + 4 \times \frac{1}{2} - \left(\frac{2}{\sqrt{3}}\right)^2 (\sqrt{3})^2$

$$= 4 \times \tfrac{1}{2} + 2 - \tfrac{4}{3} \times 3$$
$$= 2 + 2 - 4 = 0.$$

3. *Verify* $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$, *where* $A = 60^\circ$, $B = 30^\circ$.

L.H.S. $= \tan(A - B) = \tan(60^\circ - 30^\circ) = \tan 30^\circ = \dfrac{1}{\sqrt{3}}$

Again R.H.S. $= \dfrac{\tan A - \tan B}{1 + \tan A \tan B} = \dfrac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ}$

$$= \frac{\sqrt{3} - \dfrac{1}{\sqrt{3}}}{1 + \sqrt{3} \cdot \dfrac{1}{\sqrt{3}}} = \frac{\dfrac{3-1}{\sqrt{3}}}{1+1} = \frac{\dfrac{2}{\sqrt{3}}}{2} = \frac{1}{\sqrt{3}}$$

$\therefore$ L.H.S. = R.H.S.

4. *Solve the equation* $2\sqrt{3}\cos^2\theta = \sin\theta$.

In order to solve any trigonometric equation, we must try to bring the squared ratios into the form of the trigonometric ratio, which is of the first power. Here $\sin\theta$ is of first power, $\therefore$ we shall express $\cos^2\theta$ in the form of $\sin^2\theta$ by the help of the formula $\cos^2\theta = 1 - \sin^2\theta$.

$\therefore \quad 2\sqrt{3}(1-\sin^2\theta) = \sin\theta$

or $\quad 2\sqrt{3} - 2\sqrt{3}\sin^2\theta = \sin\theta$

or $\quad 2\sqrt{3}\sin^2\theta + \sin\theta - 2\sqrt{3} = 0$

$$\therefore \quad \sin\theta = \frac{-1 \pm \sqrt{1+48}}{4\sqrt{3}}$$

$$= \frac{-1 \pm 7}{4\sqrt{3}} = \frac{6}{4\sqrt{3}} \text{ or } -\frac{8}{4\sqrt{3}}$$

$$= \frac{\sqrt{3}}{2} \text{ or } -\frac{2}{\sqrt{3}}$$

(*i*) when $\sin\theta = \frac{\sqrt{3}}{2}$, $\theta = 60°$

As $\sin\theta$ is never greater than unity numerically, therefore the value $-\frac{2}{\sqrt{3}}$ is rejected.

5. *Solve* $\sin^2\theta + \frac{3}{2}\cos\theta = \frac{3}{2}$

Here $\cos\theta$ is of first degree $\therefore$ we express $\sin^2\theta$ in the form of $\cos^2\theta$.

$\therefore$ The equation can be written as

$(1-\cos^2\theta) + \frac{3}{2}\cos\theta = \frac{3}{2}$

or $\quad 2 - 2\cos^2\theta + 3\cos\theta = 3$

or $\quad 2\cos^2\theta - 3\cos\theta + 1 = 0$

$$\therefore \quad \cos\theta = \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4}$$

$$= 1 \text{ or } \tfrac{1}{2}$$

(*i*) when $\cos\theta = 1$; $\theta = 0°$

(*ii*) when $\cos\theta = \frac{1}{2}$, $\theta = 60°$

$\therefore$ the two values of θ are $0°$ and $60°$

6. *Solve* $3\sec^4\theta + 8 = 10\sec^2\theta$

Here both the *t*-ratios involved are $\sec\theta$ $\therefore$ we treat it as an equation in $\sec^2\theta$.

The equation is $3\sec^4\theta - 10\sec^2\theta + 8 = 0$

or $$(3\sec^2\theta - 4)(\sec^2\theta - 2) = 0$$

∴ $$\sec^2\theta = \tfrac{4}{3} \text{ or } 2$$

∴ $$\sec\theta = \frac{2}{\sqrt{3}} \text{ or } \sqrt{2}$$

(*i*) When $\sec\theta = \dfrac{2}{\sqrt{3}}$, $\theta = 30°$

(*ii*) When $\sec\theta = \sqrt{2}$, $\theta = 45°$

∴ the true values of θ are 30° and 45°

Note. In these examples, we have found out those values of θ which are less than a rt. ∠.

EXERCISE III (*a*)

1. Prove that $\sin 30° \cos 60° + \cos 30° \sin 60° = 1$.
2. Find the value of $\operatorname{cosec}^4 45°.\ \sec^4 30°.\ \sin^3 90°.\ \cos 60°$.
3. If $A = 30°$, verify that

(*a*) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

(*b*) $\cos 3A = 4\cos^3 A - 3\cos A$

(*c*) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

[**Sol.** (*c*) L.H.S. $= \tan 60° = \sqrt{3}$ R.H.S. $= \dfrac{2\tan 30°}{1 - \tan^2 30}$

$$= \frac{2 \times \frac{1}{\sqrt{3}}}{1 - \left(\frac{1}{\sqrt{3}}\right)^2} = \frac{2 \times \frac{1}{\sqrt{3}}}{\frac{2}{3}} = \sqrt{3} \quad \therefore \text{ L.H.S.} = \text{R.H.S.}]$$

4. When $A = 45°$, $B = 30°$, $C = 60°$; find the value of $\sin A \cos B \cos C - \cos A \sin B \sin C + \cos A \cos B \cos C$.

5. Find the value of

$$\frac{\tan 30° \tan 60° + \tan 60° \tan 45° + \tan 45° \tan 30°}{\tan 30° + \tan 45° + \tan 60°}$$

6. Prove that

$$\sin^2\frac{\pi}{6} : \sin^2\frac{\pi}{4} : \sin^2\frac{\pi}{3} : \sin^2\frac{\pi}{2} :: 1 : 2 : 3 : 4.$$

7. Prove that

$\cos^2 30° + \cos^2 60° + \cos^2 90° + 2\cos 30° \cos 60° \cos 90° = 1$.

8. When $A = 45°$ verify that

(*i*) $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$. (*ii*) $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(*iii*) $\tan A = \dfrac{\sin 2A}{1+\cos 2A}$

9. Find the value of

$$\operatorname{cosec}^2\frac{\pi}{4}\sin\frac{\pi}{6}+\operatorname{cosec}^2\frac{\pi}{3}\cos^2\frac{\pi}{6}+\sec^2\frac{\pi}{3}\cos\frac{\pi}{2}$$

Solve the following equations :—

10. $\sin^2\theta - 2\cos\theta + \frac{1}{4} = 0$.

11. $\sin\theta + \cos\theta = 1$.

[Hint : $\sin\theta = 1 - \cos\theta$ square, $\sin^2\theta = 1 + \cos^2\theta - 2\cos\theta$

or $1 - \cos^2\theta = 1 + \cos^2\theta - 2\cos\theta$ or $2\cos^2\theta + 2\cos\theta = 0$ etc.

12. $\tan\theta\sin\theta - \sin\theta = \theta$.

13. $\tan\theta + \cot\theta = 2$.

Hint : Put $\cot\theta = \dfrac{1}{\tan\theta}$ and then multiply both sides by $\tan\theta$.]

14. Given $\sin(A-B) = \dfrac{\sqrt{3}}{2}$ and $\sin(B+C) = 1$; find A and B. (A and B being positive acute angles.)

[Hint. $\therefore A - B = 60°$ and $A + B = 90°$; solve them simultaneously.]

15. Given $\tan(A-B) = \dfrac{1}{\sqrt{3}}$ and $\cos(A+B) = 0$, find A and B. (A and B being positive acute angles.)

16. Solve $\cos(2x+3y) = \frac{1}{2}$ and $\cos(3x+2y) = \dfrac{\sqrt{3}}{2}$.

(*P.U.* 1944)

SECTION II

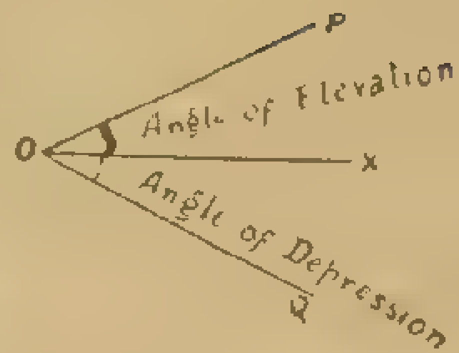
3·8. Simple cases of Heights and Distances.

The use of trigonometrical functions enables us to solve many simple and useful problems in the measurements of heights and distances. Thus Trigonometry plays a very important part in 'land survey.' By means of trigonometry astronomers have been able to calculate the distances of the sun, moon, planets and the stars.

Definition. Angle of Elevation and Depression.

When an object is observed from below, the angle which the line joining it to the observer's eye makes with the horizon is called *the angle of elevation* or the *altitude* of the object. When the observer is higher than the object, the corresponding angle is called the *angle of depression.*

In the figure O is the observer's eye, OX is a horizontal line, P and Q are any two objects in the vertical plane containing OX.



The $\angle XOP$ is the angle of elevation of the object P, and $\angle XOQ$ is the angle of depression of the object Q.

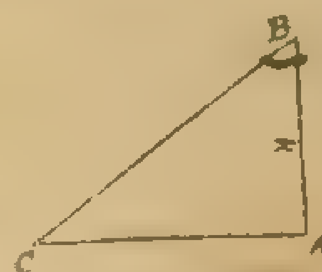
Example 1. *The angle of elevation of the top of a building is 37° 19′ 30″ observed from a point distant 362·5 feet on a horizontal plane; find its height.*

Given tan 37° 19′ 30″ = ·7624.

Let AB be the building.

Then AC = 362·5 ft.

and $\angle BCA = 37^\circ\ 19'\ 39''$.



Let AB be equal to x, then $\dfrac{x}{AC} = \tan BCA$

or $\dfrac{x}{362\cdot5} = \tan 37^\circ\ 19'\ 30'' = \cdot7624$

$\therefore \quad x = 362\cdot5 \times \cdot7624$

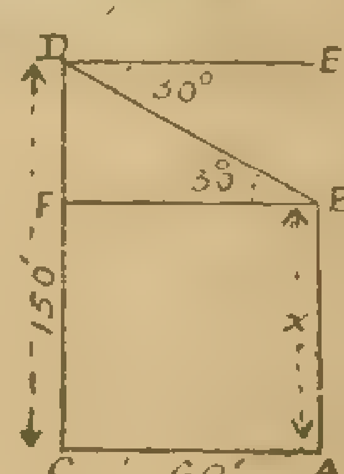
$= 276\cdot37$ ft.

Note : In order to find some side of a rt. $\angle$d. $\triangle$, the rule is $\dfrac{\text{Required Side}}{\text{Known Side}} =$ a certain *t*-ratio of the known angle.

Example 2. *The distance between two towers is 60 ft. and the angular depression of the top of the first as seen from the top of the second, which is 150 ft. high, is 30° : find the height of the first.*

Let AB and CD be the two towers and let AB be equal to x.

Then $\angle EDB = 30^\circ$, AC = 60 ft, CD = 150 ft.



From the $\angle$ BDF

we get $\dfrac{DF}{FB} = \tan 30^\circ$

or $\dfrac{CD - CF}{BF} = \dfrac{1}{\sqrt{3}}$

or $\dfrac{150 - x}{60} = \dfrac{1}{\sqrt{3}}$

or $150\sqrt{3}-x\sqrt{3}=60$

or $x\sqrt{3}=150\sqrt{3}-60$

or $x=\dfrac{150\sqrt{3}-60}{\sqrt{3}}=150-\dfrac{60}{\sqrt{3}}=150-20\sqrt{3}$

$=115{\cdot}36$ feet

Example 3 *A vertical flagstaff stands on a horizontal plane; from a point distant 150 ft. from its foot, the angle of elevation of its top is found to be 30°: find the height of the flagstaff.*

(*P.U.* 7942)

Let QP be the flagstaff and O the position of the observer.

Then OQ = 150 ft $\angle POQ=30°$

Let PQ be x ft.

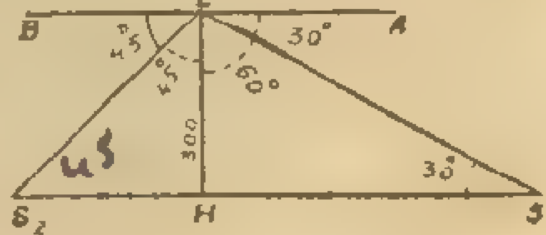
then $\dfrac{x}{150}=\tan 30°$

$=\dfrac{1}{\sqrt{3}}$

$\therefore\ x=\dfrac{150}{\sqrt{3}}=\dfrac{150\sqrt{3}}{3}$

$=50\sqrt{3}=50\times1{\cdot}732=86{\cdot}6$ ft. nearly.

Example 4. *From a lighthouse the angles of depression of two ships on opposite sides of the lighthouse are observed to be 30° and 45°. If the height of the lighthouse be 300 feet, find the distance between the ships if the line joining them passes through the foot of the lighthouse.* (*P.U.* 1941)



Let LH be the lighthouse, S_1 and S_2 the two ships Then the line S_1S_2 passes through H and LH is $\perp S_1S_2$.

Now $\angle S_1LH=60°$

$\therefore\ \angle ALS_1=30°$

and $\angle S_2LH=45°$

$\therefore\ \angle BLS_2=45°$

Let $S_1H=x$. $S_2H=300$

as $\angle HS_2L=45°$

$\dfrac{S_1H}{LH}=\tan 60°=\sqrt{3}$ or $\dfrac{x}{300}=\sqrt{3}$ $\therefore\ x=300\sqrt{3}=300\times1{\cdot}732$

$=519{\cdot}6$ ft. nearly.

$\therefore \quad S_1S_2 = 300 + 519{\cdot}6$ ft. nearly
$= 886{\cdot}6$ ft. nearly

Example 5. *The angle of elevation of a tower from a point A due south of it is x and from a point B due east of A is y. AB = l, shown that the height of the tower is given by* $h^2(\cot^2 y - \cot^2 x) = l^2$. *(P.U. 1943)*

Let h be the height of the tower PQ,

$\therefore$ A is due south of it

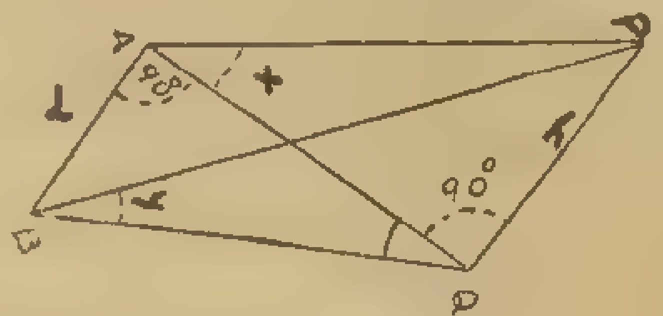
$\therefore \angle PQA = 90°$

and $\angle PAQ = x$ (given)

Again $\because$ B is due east of A

$\therefore \angle QAB =$ a right angle

and $\angle PBQ = y$.



Also $\angle PBQ = 90°$. Now from the $\triangle$ AQP, $\frac{AQ}{h} = \cot x$

$\therefore \quad AQ = h \cot x$ and from the $\triangle$ BQP, $\frac{BQ}{h} = \cot y$

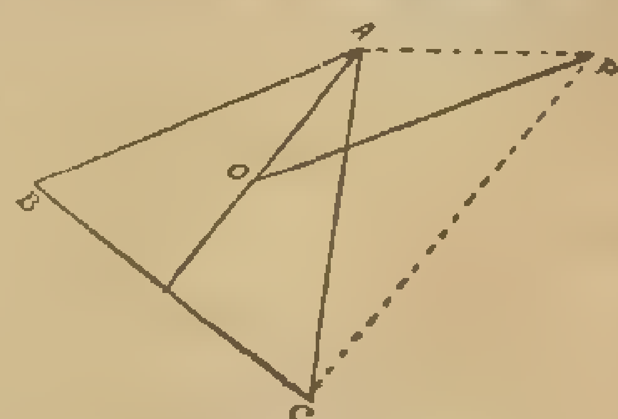
$\therefore \quad BQ = h \cot y \qquad \because \angle QAB = 90°$

$\therefore \quad BQ^2 = AQ^2 + AB^2$ or $h^2 \cot^2 y = h^2 \cot^2 x + l^2$

$\therefore \quad h^2(\cot^2 y - \cot^2 x) = l^2$

Example 6. *A pole 100 ft. high stands in the centre of an equilateral triangle which is horizontal. From the top of the pole each side subtends an angle of 60° ; prove that the length of the side of the triangle is 50 $\sqrt{6}$ ft.* (P.U. 1944)

Let ABC be the equilateral $\triangle$ in the horizontal plane and OP the pole which stands vertically at O, the centre of the triangle Each side subtends an angle of 60° at the top of the pole



$\therefore \quad \angle APC = 60°$

$\angle POA =$ a right angle

and $\because \quad AO = OC$

$\therefore \quad PA = PC$

$(\therefore \quad PA^2 = PO^2 + AO^2$ and $PC^2 = PO^2 + OC^2)$

but $\angle APC = 60°$ $\therefore \quad PA = PC = AC$

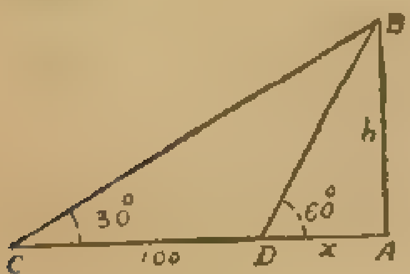
Let the side AC be equal to a, than $AD = \dfrac{a\sqrt{3}}{2}$

$$AO = \tfrac{2}{3} \times \frac{a\sqrt{3}}{2} = \frac{a}{\sqrt{3}} \quad \therefore \; PA = \sqrt{100^2 + \frac{a^2}{3}} \text{ but } AC = PA$$

$$\therefore \quad a = \sqrt{100^2 + \frac{a^2}{3}} \quad \text{or } a^2 = 100^2 + \frac{a^2}{3} \quad \text{or } \frac{2a^2}{3} = 100^2$$

or $a^2 = 5000 \times 3 = 15000 \quad \therefore \; a = 50\sqrt{6}$ ft.

Example 7. *The angle of elevation of the top of a tower is 30° from a point on the ground. On walking 100 feet towards the tower, the angle is found to be 60°. Find the height of the tower and the distance of the first pt. from the tower.* (P.U. 1934 S)



Let h be the height of the tower AB. Let C be a point where the angle of elevation is 30°. Let CD = 100. The angle of elevation ALB = 60°

Let AD be equal to x.

$$\text{Then } \frac{h}{x} = \tan 60° = \sqrt{3} \qquad \ldots (1)$$

$$\text{and } \frac{h}{x+100} \tan 30° = \frac{1}{\sqrt{3}} \qquad \ldots (2)$$

We have now to solve (1) and (2) simultaneously

$$\text{From (1) } x = \frac{h}{\tan 60°} = \frac{h}{\sqrt{3}} \qquad \ldots (3)$$

$$\text{Substituting in (2) we get } \frac{h}{h/\sqrt{3} + 100} = \frac{1}{\sqrt{3}}$$

$$\therefore \quad h = \frac{\frac{h}{\sqrt{3}} + 100}{\sqrt{3}} = \frac{h}{3} + \frac{100}{\sqrt{3}}$$

$$\text{or } \frac{2h}{3} = \frac{100}{\sqrt{3}} \quad \text{or } h = 50\sqrt{3} = 50 \times 1{\cdot}732 = 86{\cdot}6 \text{ nearly}$$

$$\text{From (3) } x = \frac{h}{\sqrt{3}} = \frac{50\sqrt{3}}{\sqrt{3}} = 50 \quad \therefore \; AC = 150 \text{ ft.}$$

EXERCISE III (b)

1. The angle of elevation of the top and the foot of a flagstaff fixed on a wall are 60° and 45°, to a man standing on

the other side of a road 30 ft. wide. Find the heights of the flagstaff and the wall.

2. AB is a straight shore 1366 ft. long aud C is a point at sea. If the angles A and B are 30° and 45° respectively, find the distance of C from AB. (*P. U. 1934*)

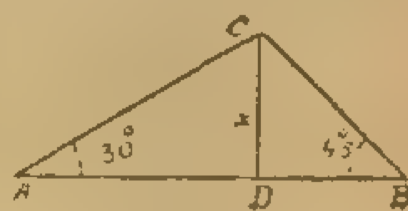
[Hint: Let $CD=x$

Then $\dfrac{AD}{x}=\cot 30^\circ=\sqrt{3}$

$\therefore \quad AD=\sqrt{3}x$

Also $\dfrac{BD}{x}=\tan 45^\circ=1$

$\therefore \quad \sqrt{3}x+x=1366$ etc.]



3. On the straight coast there are three objects A, B, C, such that $AB=BC=2$ miles. A vessel approaches B in a line perpendicular to the coast at a certain point AC is found to subtend an angle of 60°, after sailing in the same direction for 10 minutes AC is found to subtend 120°; find the rate at which the ship is going. (*C.U. 1912*)

4. The angular elevation of a tower at a place A due south of it is θ°, and at a place B due west of A and at a distance 'a' from it the elevation is ϕ°. Find the height of the tower. (*C.U. 1910*)

5. A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°; when he retires 45 feet from the bank he finds the angle to be 30°. Find the height of the tree and the breadth of the river.

(*P U. 1942-S*)

6. From the top of a cliff, 200 ft. high, the angles of depression of the top and bottom of a tower are observed to be 30° and 60°. Find the height of the tower. (*D.U.*)

7. Two parts of the same height stand on either side of a road 120 ft. wide: at a point in the road between the posts the elevation of the tops of the pillars are 60° and 30°. Find the height of the posts and the position of the point.

8. A ladder 20 ft. long reaches to a distance of 20 ft. from the top of a flag-staff. At the foot of the ladder the elevation of the top of the staff is 60°. Find the height of the flagstaff. (*P.U. 1940*)

9. From a point 100 feet above the surface of a lake, the angular elevation of a peak is 30° and the angle of

depression of the reflection of the peak is 60°. Find the height of the peak. (*P.U. 1939*)

[**Hint**: It is assumed that the depth of the reflection = the height of the peak above the surface.]

10. An observer in a boat is being rowed away from a cliff 150 feet high and it takes 3 minutes for the angle of elevation of the top of the cliff to change from 60° to 33°. Find the speed of the boat, if the water is not flowing but is at rest.

[**Hint**: Find the distance from one position of the boat to the other; that distance is covered in 3 minutes, which gives the speed.]

Miscellaneous Exercises on Chapter III

1. If $A = 430°$, verify that

(*i*) $\sin 3A = 3 \sin A - 1 \sin^3 A$

(*ii*) $\cos 3A = 4 \cos^3 A - 3 \cos A$

(*iii*) $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$.

2. By taking $\alpha = 30°$ and $\beta = 60°$ verify that

(*i*) $\sin(\alpha + \beta) \neq \sin \alpha + \sin \beta$.

and (*ii*) $\cos(\beta - \alpha) \neq \cos \beta - \cos \alpha$.

3. (*i*) Find x, if $x = \dfrac{\tan^2 45° - \tan^2 30}{\sin 30 . \cos 30 . \tan 30}$.

(*ii*) Find x, y, z (all being positive acute ∠s) if

$\sin(x + y - z) = 1$

$\cos(x - y + z) = 1$

$\tan(-x + y + z) = 1$.

4. If AL is the altitude of the △ ABC, prove that

$$AL = \frac{BC}{\cot B + \cot C}.$$ (*P.U. 1937*)

[**Hint**: $\dfrac{BL}{AL} = \cot B$ and $\dfrac{LC}{AL} = \cot C$.

But $BL + LC = BC$].

5. A man stands at a point A on the bank AB of a straight river and observes that the line joining A to a post C on the opposite bank with AB makes an angle of 30°. He then goes 200 yards along the bank to B and finds that BC makes an angle of 60° with the bank. Find the breadth of the river.

6. There are two towers on a horizontal plane. The angle of elevation of the top of the second as observed from the foot

of the first is 60° ; when observed from the foot of the second, the angle of elevation of the top of the first is 30°. Prove that the second tower is three times as high as the first.

7. The upper part of a tree broken by the wind makes an angle of 60° with the ground, the distance from the root to the point, where the top of the tree touches the ground, is 30 feet. What was the height of the tree ?

[**Hint** : Let OPQ be the height 'h' of the tree. It breaks at P and the highest top Q touches the horizantol plane at R, where OR = 30 ft. Now OP = OR tan 60°, PQ = PR = OR sec 60°, h = OP + PQ.]

8. The angle of elevation of two aeroplanes (one passing vertically over the other) are seen by an observer to be 30° and 60° respectively. If the height of the upper aeroplane above the lower aeroplane is 300 ft., find the height of the upper aeroplane above the ground.

ANSWERS TO EXERCISES ON CHAPTER III

Exercise III (a)

2. $1\frac{1}{3}$. 4. $\frac{1}{4}$, $\frac{\sqrt{3}}{\sqrt{2}}$. 5. 1. 9. 2.

10. $\frac{\pi}{3}$. 11. 0° or 90°. 12. $0, \pi, \frac{\pi}{4}$.

13. $\frac{\pi}{4}$. 14. A = 75° ; B = 15°. 15. A = 60°, B = 30°.

16. $x = -6°$; $y = 24°$.

Exercise III (b)

1. 22 ft. nearly ; 30 ft. 2. 683 ($\sqrt{3}-1$) ft.

3. $8\sqrt{3}$ miles per hour. 4. $\dfrac{a \tan\theta \tan\phi}{\sqrt{\tan^2\theta - \tan^2\phi}}$.

5. 34·64 ft. nearly ; 20 ft. 6. $132\frac{1}{3}$ ft.

7. $30\sqrt{3}$ ft. ; 90 ft. 8. 30 ft.

9. 200 ft. 10. $2000\sqrt{3}$ ft. per hour.

Miscellaneous Exercises on Chapter III

3. (i) $x = \frac{1}{8}$. (ii) $x = 45°$; $y = 67\frac{1}{2}°$; $z = 22\frac{1}{2}$.

5. $50\sqrt{3}$ yards. 7. $30(3+\sqrt{3})$ ft.

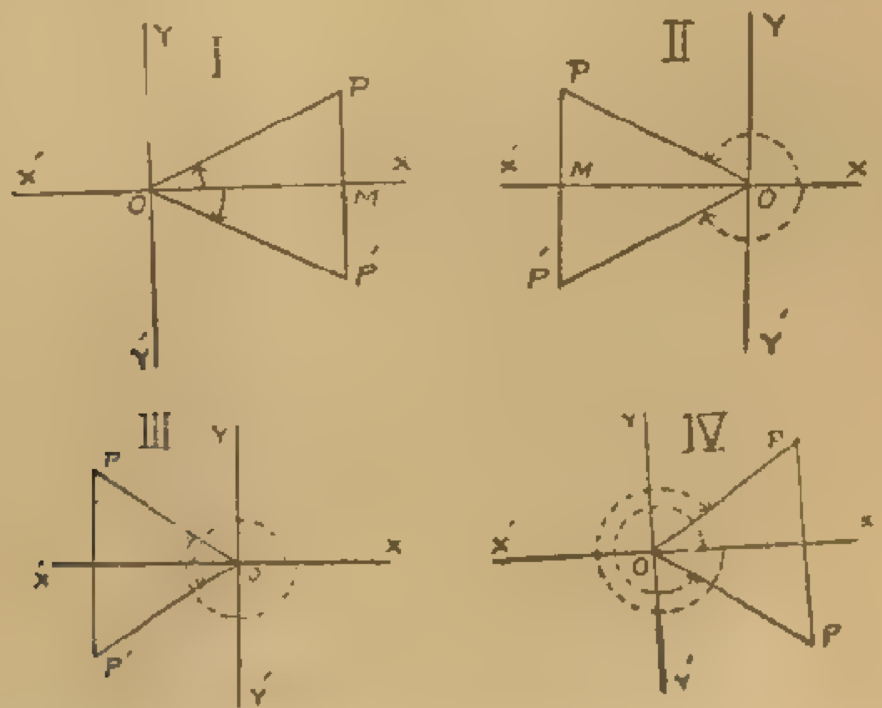
8. 6000 ft.

CHAPTER IV

TRIGONOMETRIC RATIOS ON CERTAIN ALLIED ANGLES

Def. Allied ∠s : The angles $-\theta$, $90^\circ-\theta$, $90^\circ+\theta$, $180^\circ-\theta$ etc. are called angles a allied to θ.

4·1. To find the t-ratios of $(-\theta)$ in terms of those of θ, for all values of θ.



Let the revolving line OP, starting from OX, trace out any angle XOP$=\theta$.

To obtain the angle $(-\theta)$ let the revolving line starting from the same position revolve through an angle of the same magnitude in the opposite direction and take up the position OP′ so that XOP′$=-\theta$.

Draw PM ⊥ XOX′ and produce it to meet OP′ in P′.

Then in △s POM and P′OM, we have (∠XOP may be in any quadrant)

∠POM = ∠P′OM
∠PMO = ∠P′MO = a right angle
OM = OM

$\therefore$ Δs are congruent.

Hence, having regard to the signs of lines, we have

$$MP' = -MP$$
$$OP' = OP$$
$$OM = OM$$

[**Note** : The revolving line OP is taken to be equal to OP in all its positions.]

$$\therefore \sin(-\theta) = \sin \angle XOP' = \frac{MP'}{OP'} = -\frac{MP}{OP} = -\sin\theta$$

$$\cos(-\theta) = \cos \angle XOP' = \frac{OM}{OP'} = \frac{OM}{OP} = \cos\theta$$

$$\tan(-\theta) = \tan \angle XOP' = \frac{MP'}{OM} = -\frac{MP}{OM} = -\tan\theta.$$

$$\cot(-\theta) = \cot \angle XOP' = \frac{OM}{OP'} = -\frac{OM}{MP} = -\cot\theta.$$

$$\sec(-\theta) = \sec \angle XOP' = \frac{OP'}{OM} = \frac{OP}{MP} = \sec\theta.$$

$$\operatorname{cosec}(-\theta) = \operatorname{cosec} \angle XOP' = \frac{OP'}{PM} = -\frac{OP}{MP} = -\operatorname{cosec}\theta.$$

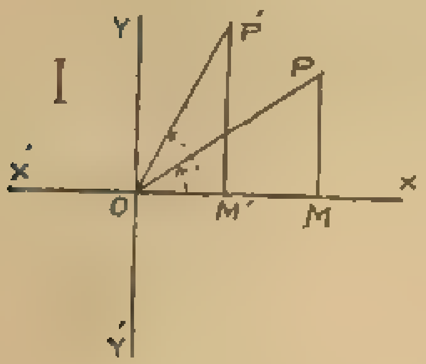
42. To find the values of trigonometrical ratios of $(90° - \theta)$ in terms of those of angle θ, for all values of θ.

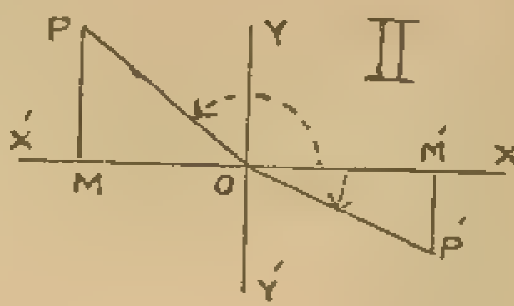
Let the revolving line starting from OX trace out any angle XOP, denoted by θ in any quadrant

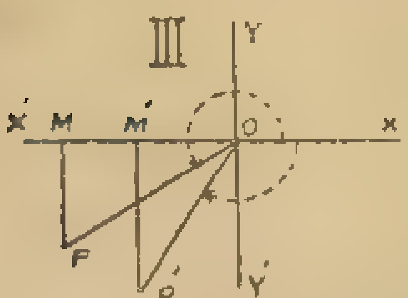
To obtain the angle $(90° - \theta)$: Let the revolving line start from OX and rotate to OY tracing out an $\angle XOY = 90°$ and then let it rotate from Y in the opposite direction through the angle of magnitude θ, and let the final position of the revolving line be OP' ; then $\angle XOX' = 90° - \theta$, which is positive if the angle θ is in the first quadrant but is negative if angle θ lies in the remaining three quadrants.

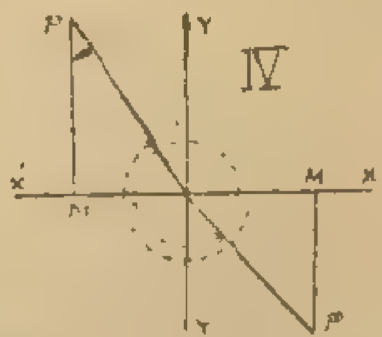
Take OP' = OP and draw perpendicluars PM and P'M' to XOX'.

Then whatever be the size of the angle θ, we shall prove that Δs POM and P'OM' are congruent.









(*i*) See Fig. (*i*) when $\angle\theta < 90°$. (*i.e.*) θ lies in the quadrant.

$$OP = OP'$$
$$\angle POM = \angle\theta = \angle YOP' = \angle OP'M'$$
$$\angle PMO = \angle P'M'O = 90° \text{ (each)}$$

$\therefore$ Δs are congruent ($\equiv$)

(*ii*) Fig. (*ii*) when $90° < \theta < 180°$ *i.e.* θ lies in the second quadrant.

$$OP = OP' \; ; \; \angle PMO = \angle P'M'O$$

and $\angle POM = 180° - \theta = 180° - YOP'$

($\because$ $\theta = \angle YOP'$ by construction)

$= \angle Y'OP' = \angle OP'M'$ (alt. $\angle$s).

$\therefore$ Δs are $\equiv$.

(*iii*) Fig. (*iii*) when $180° < \theta < 270°$ (*i.e.*) θ lies in the third quadrant.

$$OP = OP' \; ; \; \angle PMO = \angle P'M'O$$

and $\angle POM = \theta - 180° = YOP' - 180°$

($\because$ $\theta = YOP'$ by construction)

$= P'OY' = OP'M'$

$\therefore$ Δs are $\equiv$.

(*iv*) Fig. (*iv*) when $270° < \theta < 360°$ (*i.e.*) θ lies in the fourth quadrant.

$$OP = OP' \; ; \; \angle PMO = \angle P'M'O$$

$\angle POM = 360° - \theta = 360° - (YOP')$ where YOP' is

measured in the negative direction.

$$= P'OY = OP'M'$$

$\therefore$ Δs are $\equiv$.

Hence having regard to signs of lines, we get

$$OM' = MP\ ;\ M'P' = OM \text{ and } OP = OP.$$

$$\therefore \quad \sin (90-\theta) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\theta$$

$$\cos (90-\theta) = \frac{OM'}{OP'} = \frac{MP}{OP} = \sin\theta$$

$$\tan (90^\circ-\theta) = \frac{M'P'}{OM'} = \frac{OM}{MP} = \cot\theta$$

$$\cot (90^\circ-\theta) = \frac{OM'}{M'P'} = \frac{MP}{OM} = \tan\theta$$

$$\sec (90^\circ-\theta) = \frac{OP'}{OM'} = \frac{OP}{MP} = \operatorname{cosec}\theta$$

$$\operatorname{cosec} (90^\circ-\theta) = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta.$$

Note (*i*) ; *How to draw the four figures.*

Take an angle $\theta = \alpha^\circ$ (or 30°), 120°, 210°, 300° we get the positions of OP ; and by taking $90-\alpha = 60^\circ$, or $90-120 = -30^\circ$, $90-210^\circ = -120^\circ$ and $90^\circ-300^\circ = -210^\circ$, we get the positions of OP′.

Note (*ii*) : The examinees should note that if in the examination they are asked to prove the various ratios when θ lies in a particular quadrant, they should draw the relevant figure and not all the figures, *e.g.*, if θ lies in the second quadrant, *i.e.*, $90^\circ < \theta < 180^\circ$, the second figure is to be drawn and proof for this figure is to be given. If however, the question is set for all values of θ, all the four figures are to be drawn.

4·3r. To find the trigonometrical ratios of the angle $(90^\circ+\theta)$ in terms of those of the angle θ, for all values of θ.

Let the revolving line starting form OX trace out any angle $XOP = \theta$ in any quadrant.

To obtain the angle $(90^\circ+\theta)$, let the revolving line rotate further from the position OP through a right angle in the

positive direction to the position OP′. Then $\angle XOP' = 90^\circ + \theta$. It is positive in all the quadrants.

Take $OP' = OX$ and draw PM and P′M′ perpendiculars to XOX′.

Fig. (*i*) when $0 < \theta < 90^\circ$. $OP' = OP$; $\angle OM'P' = \angle OMP$
$= \text{a rt. } \angle$.

$$\angle P'OM' = 180 - (90 + \theta) = 90 - \theta = \angle OPM$$

$\therefore$ $\triangle$s are $\equiv$

Fig. (*ii*) when $90 < \theta < 180^\circ$; $O'P' = OP$; $\angle OM'P' = \angle OMP$.

$$\angle P'OM' = (90 + \theta - 180) = \theta - 90^\circ = \angle POY = \angle OPM.$$

$\therefore$ $\triangle$s are $\equiv$

Fig. (*iii*) when $180 < \theta < 270^\circ$; $OP' = OP$; $\angle OM'P' = \angle OMP$.

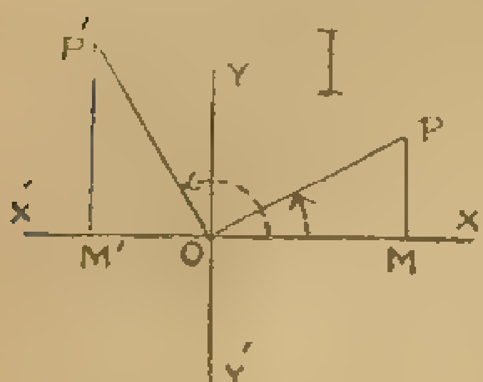
$$\angle P'OM' = [360^\circ - (90 + \theta)] = 270^\circ - \theta = \angle POY'$$
$$= \angle OPM.$$

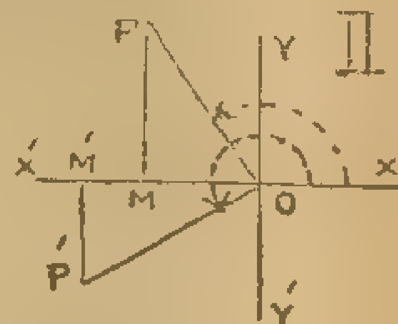
$\therefore$ $\triangle$s are $\equiv$

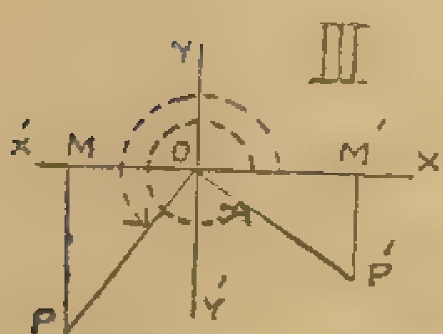
Fig. (*iv*) when $270 < \theta < 360^\circ$ $OP' = OP$; $\angle OM'P' = \angle OMP$.

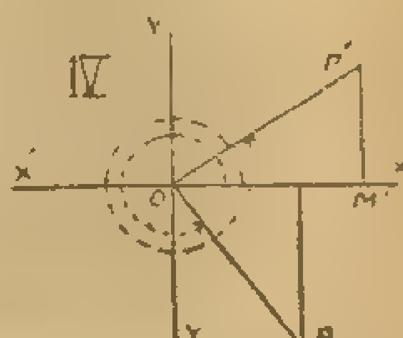
$$\angle P'OM' = (\theta + 90 - 360^\circ) = (\theta - 270) = \angle POY'$$
$$= \angle OPM.$$

$\therefore$ $\triangle$s are $\equiv$.









Hence with the due regard to signs of lines, we have

$OM' = -MP$ [$\because$ Both do not agree in sign.]

$M'P' = OM$ [$\because$ Both agree in sign.]

$OP' = OP$

$$\therefore \quad \sin(90^\circ+\theta)=\frac{M'P'}{OP'}=\frac{OM}{OP}=\cos\theta. \qquad (P.U.\ 1948)$$

$$\cos(90^\circ+\theta)=\frac{OM'}{OP'}=-\frac{MP}{OP}=-\sin\theta. \qquad (P.U.\ 1949)$$

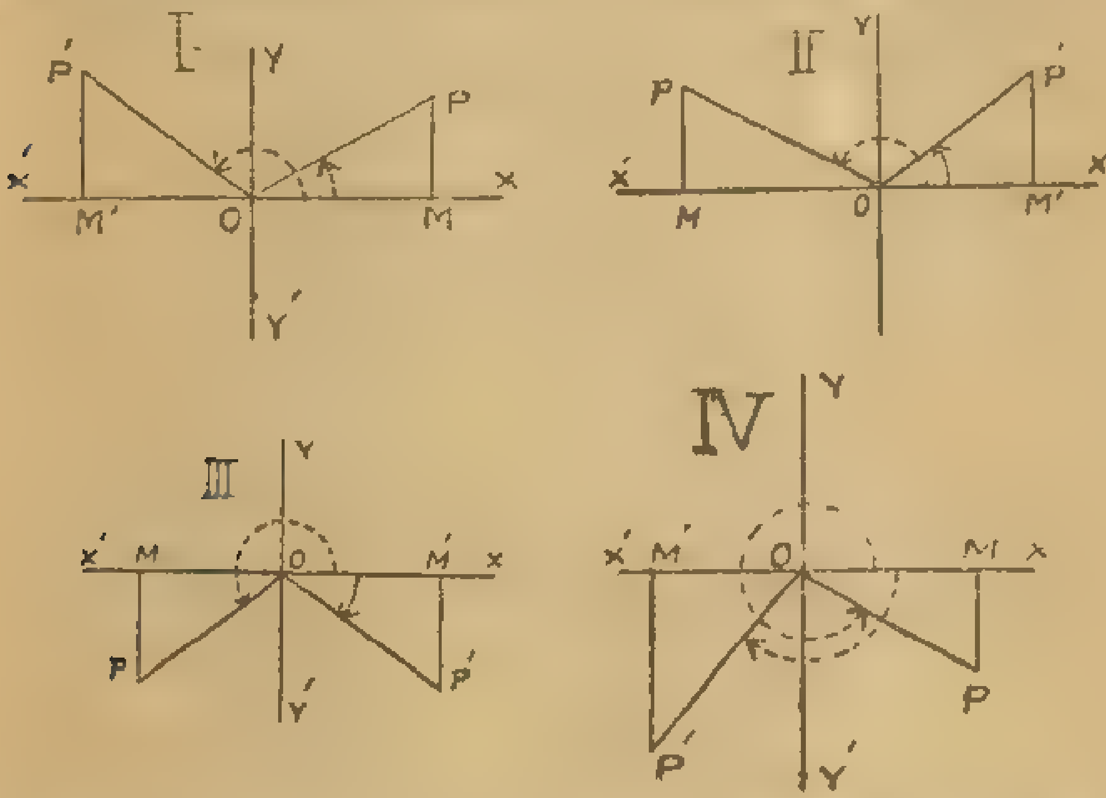
$$\tan(90^\circ+\theta)=\frac{M'P'}{OM'}=\frac{OM}{-MP}=-\cot\theta$$

$$\cot(90^\circ+\theta)=\frac{OM'}{M'P'}=\frac{-MP}{OM}=-\tan\theta$$

$$\sec(90^\circ+\theta)=\frac{OP'}{OM'}=\frac{OP}{-MP}=-\operatorname{cosec}\theta$$

$$\operatorname{cosec}(90^\circ+\theta)=\frac{OP'}{M'P'}=\frac{OP}{OM}=\sec\theta.$$

4·4. To find the trigonometrical ratios of the angle $(180^\circ-\theta)$ in terms of those of θ for all values of θ.



Let the revolving line starting from OX trace out any angle $XOP=\theta$ in any quadrant. Now to obtain the angle $(180-\theta)$, let the revolving line first coincide with OX′ and then revolve in the clockwise direction (*i.e.*, opp. direction) through θ and take up the position of OP′; then $\angle XOP = 180-\theta$, which is $+$ve if the $\angle\theta$ is in the first two quadrants and is $-$ve if the $\angle\theta$ is in the third and fourth quadrants.

Take $OP' = OP$. Draw $P'M'$ and PM $\perp$s to XOX'.

(*i*) when $0° < \theta < 90°$: fig. (*i*)

$OP' = OP$; $\angle OM'P' = \angle OMP$;

$\angle P'OM' = 180° - (180° - \theta) = \theta = \angle POM$.

$\therefore$ $\triangle$s are $\equiv$.

(*ii*) when $90° < \theta < 180°$: fig. (*ii*)

$$\left.\begin{array}{l} OP' = OP \text{ ; } \angle OM'P' = \angle OMP \text{ ;} \\ \angle P'OM' = 180° - \theta = \angle POM \end{array}\right\} \quad \therefore \quad \triangle\text{s are } \equiv$$

(*iii*) when $180° < \theta < 270°$

$$\left.\begin{array}{l} OP' = OP \text{ ; } \angle OM'P' = \angle OMP \\ \angle P'OM' = \theta - 180° = 180° + POM - 180° \\ \qquad = \angle POM \end{array}\right\} \quad \therefore \quad \triangle\text{s are } \equiv$$

(*iv*) when $270° < \theta < 360°$

$OP' = OP$; $\angle OM'P' = OMP$

and $\angle P'OM' = 180° - MOP' = 180 - (\theta - 180)$

$= 360° - \theta = \angle POM$

$\therefore$ $\triangle$s are $\equiv$.

Hence, having regard to signs of lines, we have

$OM' = -OM$ [$\because$ Both do not agree in sign.]

$M'P' = MP$ [$\because$ Both agree in sign.]

$OP' = OP$

$$\therefore \quad \sin(180° - \theta) = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin$$

$$\cos(180° - \theta) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos\theta$$

$$\tan(180° - \theta) = \frac{M'P'}{OM'} = \frac{MP}{-OM} = -\tan\theta$$

$$\cot(180° - \theta) = \frac{OM'}{M'P'} = \frac{-OM}{MP} = -\cot\theta$$

$$\sec(180° - \theta) = \frac{OP'}{OM'} = \frac{OP}{-OM} = -\sec\theta$$

$$\operatorname{cosec}(180° - \theta) = \frac{OP'}{M'P'} = \frac{OP}{MP} = \operatorname{cosec}\theta.$$

Note (*i*) *Supplementary and Complementary Angles.*

(*i*) If the sun of two $\angle$s is 180°, they are called Supplementary $\angle$s, *e.g.* ; 150° and 30° are Supplementary Angles ; each called the Supplement of the other.

(*ii*) If the sum of two ∠s is 90°, they are called Complementary ∠s, *e g.*, 30° and 60° are complementary angles and each is called the Complement of the order.

Note (*ii*) How to draw the four figures.

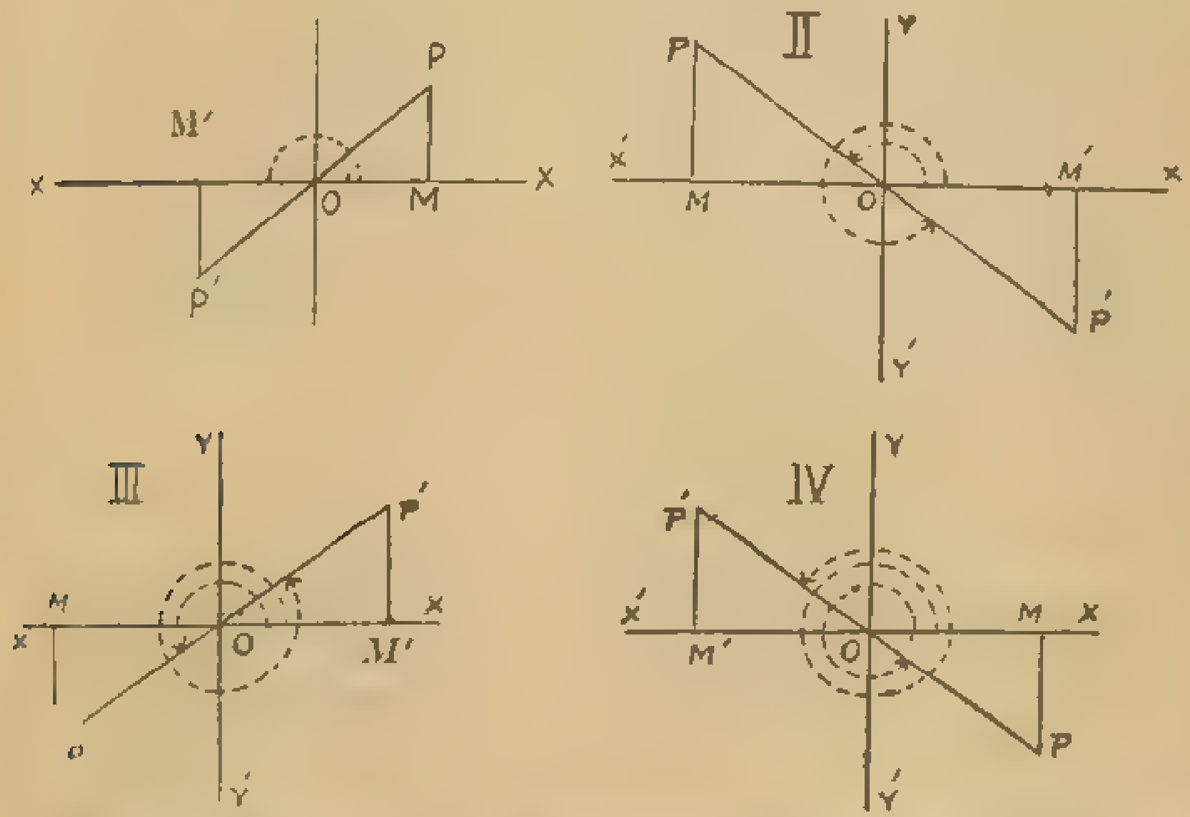
Take $\theta=30°$, 120°, 210° and 300°, this will give the positions of OP respectively ; and correspondingly by taking the angles $(180-30)=150°$, $(190-120)=60°$, $(180-210)=-30°$ and $(180-300)=-120°$, we shall get the positions of OP′.

Note (*iii*) These results could also be proved as follows :

$$\sin(180-\theta)=\sin(90+\overline{90-\theta})=\sin(90+A),\text{ where } A=\overline{90-\theta})$$
$$=+\cos A=+\cos(60-\theta)=\sin\theta$$

$\cos(182=\theta)=\cos(90+\overline{90-\theta})=-\sin(90-\theta)=-\cos\theta$ and so on.

4 5. To find the trigonometric ratios of the angles $(180°+\theta)$ in terms of those of the angle θ, for all values of θ.



Let the revolving line starting from OX trace out an angle $XOP=\theta$ in any quadrant.

To obtain the angle $(180°+\theta)$, let the revolving line revolves further from OP in the positive direction to OP′ through two

right angles so that $\angle XOP' = 180^\circ + \theta$. It is positive in all the quadrants.

Take $OP' = OP$. Draw PM and P′M′ perpendiculars to XOX′.

(*i*) Fig. (*i*) when $0 < \theta < 90^\circ$, $OP' = OP$; $\angle OM'P' = \angle OMP$;
$\angle P'OM' = 180^\circ + \theta - 180^\circ = \theta = \angle POM$.
$\therefore$ $\triangle$s are $\equiv$.

(*ii*) Fig. (*ii*) when $90 < \theta < 180^\circ$, $OP' = OP$; $\angle OM'P' = \angle OMP$;
$\angle P'OM' = 360^\circ - (180^\circ + \theta) = 180 - \theta = \angle POM$.
$\therefore$ $\triangle$s are $\equiv$.

(*iii*) Fig. (*iii*) when $180^\circ < \theta < 270^\circ$: $OP' = OP$; $\angle OM'P' = \angle OMP$;
$\angle P'OM' = (180^\circ + \theta - 360^\circ) = \theta - 180 = \angle POM$.
$\therefore$ $\triangle$s are $\equiv$.

(*iv*) Fig. (*iv*) when $270 < \theta < 360^\circ$. $OP' = OP$; $\angle OM'P' = \angle OMP$
$\angle P'OM' = (360^\circ + 180^\circ - 180 + \theta)$
$= 360^\circ - \theta = \angle POM$. $\therefore$ $\triangle$s are $\equiv$.

Hence, having regard to signs of lines, we have
$OP' = OP$; $OM' = -OM$ and $P'M' = -MP$.
[$\because$ they do not agree in sign in all figures]

$$\therefore \quad \sin(180^\circ + \theta) = \frac{M'P'}{OP'} = \frac{-MP}{OP} = -\sin\theta$$

$$\cos(180^\circ + \theta) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos\theta$$

$$\tan(180^\circ + \theta) = \frac{M'P'}{OM'} = \frac{-MP}{-OM} = \tan\theta$$

$$\cot(180^\circ + \theta) = \frac{OM'}{M'P'} = \frac{-OM}{-MP} = \cot\theta$$

$$\sec(180^\circ + \theta) = \frac{OP'}{OM'} = \frac{OP}{-OM} = -\sec\theta$$

$$\operatorname{cosec}(180^\circ + \theta) = \frac{OP'}{M'P'} = \frac{OP}{-MP} = -\operatorname{cosec}\theta.$$

Note. These results could also be obtained as follows :

$\sin(180 + \theta) = \sin(90 + 90 + \theta) = \sin(90 + A)$, where
$A = 90 + \theta = \cos A = \cos(90 + \theta) = -\sin\theta$.

Similarly $\cos(180^\circ + \theta) = \cos(90^\circ + 90^\circ + \theta)$
$= \cos(90^\circ + A) = -\sin A$
$= -\sin(90^\circ + \theta) = -\cos\theta$

$\tan(180^\circ + \theta) = \tan(90^\circ + 90^\circ + \theta) = \tan(90^\circ + A)$
$= -\cot A = -\cot(90 + \theta) = \tan\theta$.

4·6. To find the trigonometrical ratios of an angle $(360°+\theta)$ in terms of those of θ, for all values of θ.

In whatever position the revolving line may be when it has described an angle θ, it will be in exactly the same position when it has made one more complete revolution in the positive direction, *i.e.*, when it has described an angle $360°+\theta$.

Hence the trigonometrical ratios for an angle $360°+\theta$ are the some as those for θ.

It is clear that the ratios of the angle $(360°-\theta)$ are the same as those of the angle $(-\theta)$ since the revolving line end in the same position for both.

Also that the addition or subtraction of any multiple of $360°$ to an angle leaves the ratios unaltered.

For example $\sin(\pm n\ 360°+\theta)=\sin\theta$ where n is any integer and similarly for other trigonometrical ratios.

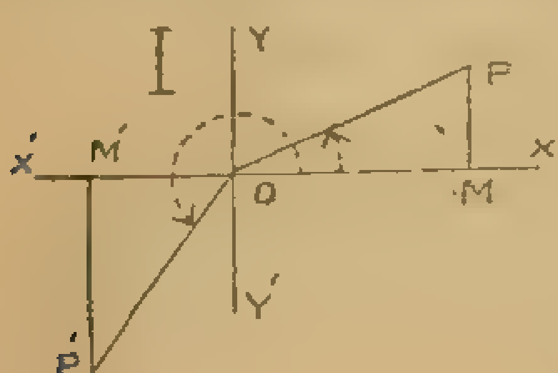
4·7. Trigonometrical relations between the angles $270°-\theta$ and θ.

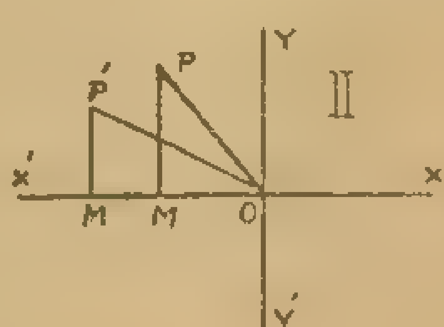
Let the revolving line starting from OX trace out any angle $XOP=\theta$ in any quadrant.

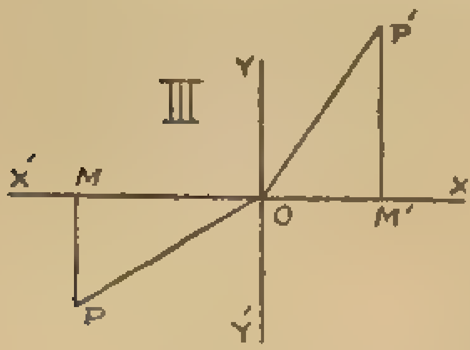
To obtain an angle $(270°-\theta)$ let the revolving line revolve up to OY′, tracing an $\angle$ of $270°$ and then rotate back through an angle θ and take up the position OP′, then $\angle XOP'=270°-\theta$. It is positive in the first, second and third quadrants and negative in the fourth quadrant.

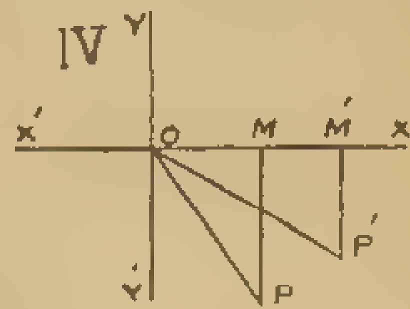
Take $OP'=OP$ and draw PM and P′M′ perpendiculars to X′OX.

Then whatever be magnitude of the angle θ, we have Δs OPM and OP′M′ congruent (various cases having been left to the students as an exercise).









$\angle POM = \theta = \angle Y'OP' = \angle OP'M'$
$\angle OMP = \angle OM'P' =$ a right angle
$OP' = OP$ $\therefore$ Δs are $\cong$

Hence, having regard to signs of lines, we have

$OM' = -MP$ [$\because$ they do not agree in sign]
$M'P' = -OM$ [$\because$ " "]
$OP' = OP$

$$\therefore \sin(270^\circ - \theta) = \frac{M'P'}{OP'} = \frac{-OM}{OP} = -\cos\theta$$

$$\cos(270^\circ - \theta) = \frac{OM'}{OP'} = \frac{MP}{OP} = -\sin\theta$$

$$\tan(270^\circ - \theta) = \frac{M'P'}{OM'} = \frac{-OM}{-MP} = \cot\theta$$

$$\cot(270^\circ - \theta) = \frac{OM'}{M'P'} = \frac{-MP}{-OM} = \tan\theta$$

$$\sec(270^\circ - \theta) = \frac{OP'}{OM'} = \frac{OP}{MP} = -\frac{OP}{MP} = -\operatorname{cosec}\theta$$

$$\operatorname{cosec}(270^\circ - \theta) = \frac{OP'}{M'P'} = \frac{OP}{-OM} = \frac{-OP}{OM} = -\sec\theta.$$

Note. Alternative Method without reference to figures.

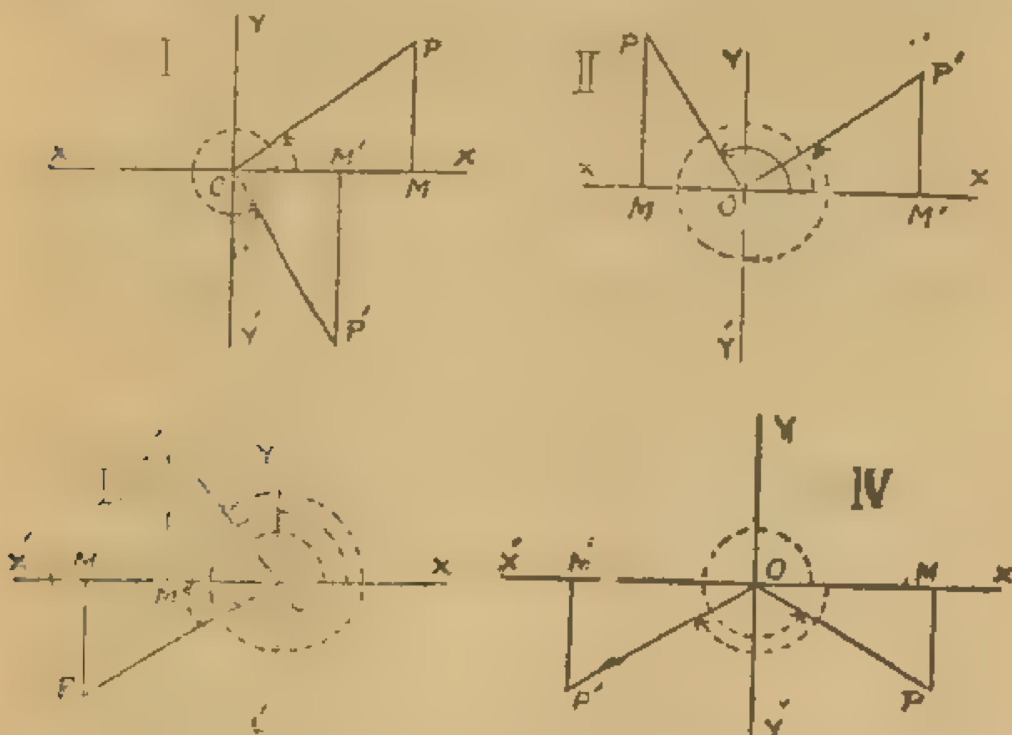
$\sin(270+\theta) = \sin(180+90-\theta) = -\sin(90-\theta) = -\cos\theta$
$\cos(270-\theta) = \cos(180+90-\theta) = -\cos(90-\theta) = -\sin\theta$
$\tan(270-\theta) = \tan(180+90-\theta) = \tan(90-\theta) = \cot\theta$

and so on.

4·8. Trigonometrical relations between the angles θ and $270^\circ+\theta$.

Let the revolving line starting from OX trace out any angle $XOP = \theta$ in any quadrant.

To obtain the angle $(270°+\theta)$, let the revolving line revolve up to OY′ tracing out an $\angle 270°$ and rotate further in the positive direction through an angle θ. Let the final position of the revolving line be OP′, then $XOP' = 270° + \theta$. It is positive in all the quadrants.



Take OP′ = OP. Draw PM and P′M′ perpendiculars to XOX′ then whatever be the magnitude of the angle θ, $\triangle$s OPM and OP′M′ are congruent.

$POM = \theta = \angle Y'OP' = \angle OP'M'$
$\angle OMP = \angle OM'P' =$ a right angle
$OP' = OP \quad \therefore \quad \triangle$s are $\cong$

Hence, having regard to signs of lines, we have

$OM' = MP$ [$\because$ they agree in sign]
$M'P' = -OM$ [$\because$ they do not agree in sign]
$OP' = OP$

$$\therefore \quad \sin(270°+\theta) = \frac{M'P'}{OP'} = \frac{-OM}{OP} = -\cos\theta$$

$$\cos(270°+\theta) = \frac{OM'}{OP'} = \frac{MP}{OP} = \sin\theta$$

$$\tan(270°+\theta) = \frac{M'P'}{OM'} = \frac{-OM}{MP} = -\cot\theta$$

$$\cot(270^\circ+\theta)=\frac{OM'}{M'P'}=\frac{MP}{-OM}=-\tan\theta$$

$$\sec(270^\circ+\theta)=\frac{OP'}{MP}=\frac{OP}{MP}=\operatorname{cosec}\theta$$

$$\operatorname{cosec}(270^\circ+\theta)=\frac{OP'}{P'M'}=\frac{OP}{-OM}=-\sec\theta.$$

Alternative Method. Without the help of figures.

$\sin(270^\circ+\theta)=\cos\ 180^\circ+\overline{90^\circ+\theta})$

Put $90^\circ+\theta=A$, then $\sin(270^\circ+\theta)=\sin(180^\circ+A)$

$$\begin{aligned}&=-\sin A\\&=-\sin(90^\circ+\theta)\\&=-\cos\theta\end{aligned}$$

$$\begin{aligned}\cos(270^\circ+\theta)&=\cos(180^\circ+90+\theta)\\&=\cos(180^\circ+A)=-\cos A\\&=-\cos(\overline{90^\circ+\theta})=+\sin\theta\end{aligned}$$

$$\begin{aligned}\tan(270^\circ+\theta)&=\tan(180^\circ+\overline{90^\circ+\theta})=\tan(180^\circ+A)=\tan A\\&=\tan(\overline{90^\circ+\theta})=-\cot\theta\end{aligned}$$

$$\begin{aligned}\cot(270^\circ+\theta)&=\cot(180^\circ+\overline{90+\theta})\\&=\cot(180+A)-\cot A=\cot(90^\circ+\theta)\\&=-\tan\theta\end{aligned}$$

$$\begin{aligned}\sec(270^\circ+\theta)&=\sec(180^\circ+\overline{90^\circ=\theta})=\sec(180^\circ+A)\\&=-\sec A=-\sec(90^\circ+\theta)\\&=\operatorname{cosec}\theta\end{aligned}$$

$$\begin{aligned}\operatorname{cosec}(270^\circ+\theta)&=\operatorname{cosec}(180^\circ+\overline{90^\circ+\theta})\\&=\operatorname{cosec}(180^\circ+A)=-\operatorname{cosec}A\\&=-\operatorname{cosec}(90^\circ+\theta)\\&=-\sec\theta\end{aligned}$$

Note. How to draw four figres. It is left to the student as an exercise.

4·9. Periods of Trigonometrical functions.

We have proved that $\sin(\theta+2\pi)=\sin\theta$

$$\begin{aligned}\cos(\theta+2\pi)&=\cos\theta\\\sec(\theta+2\pi)&=\sec\theta\\\operatorname{cosec}(\theta+2\pi)&=\operatorname{cosec}\theta\end{aligned}$$

i.e., if 2π is added to θ, the values of $\sin\theta$, $\cos\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$ do not change. This is expressed by saying that

2π is the period of $\sin\theta$, $\cos\theta$, and $\operatorname{cosec}\theta$ and these functions are periodic

Again $\because \tan(\theta+\pi)=\tan\theta$, $\cot(\theta+\pi)=\cot\theta$.

$\therefore$ π is the period of $\tan\theta$ and $\cot\theta$, and these functions are periodic.

Hence all the circular functions are periodic, the period being 2π, except for the tangent and the cotangent in which case the period is π. We, therefore, give now the defnition of periodic function.

Def. A function is said to be *periodic* when its value is constantly repeated after definite interval in the same order and 'the definite interval' is called the *period*.

4·10. Two important rules for committing to memory the functions of allied angles of θ in terms of those of θ.

(1) Any trigonometrical ratio of $180°\pm\theta$, $360\pm\theta$ is equal numerically to the same ratio of θ, and that the sign to be prefixed is the same whatever the value of θ; this sign can be determined by considering the particular case in which θ is acute and considering in which quadrant the rotating arm of the angle would lie in this particular case, *e.g.*, (*i*) $\tan(180+\theta)$ lies in the third quadrant and hence its sign shall be +ve: (*ii*) $\cos(180-\theta)$ lies in the second quardrant and hence is −ve.

(2) Any trigonometrical ratio of $90°\pm\theta$, $270°\pm\theta$ is equal numerically to the co-ratio of θ and the sign to be prefixed is the same whatever the value of θ; this sign can be determined by considering the particular case in which θ is acute and considering in which quadrant the rotating arm of the angle would lie in this particular case, *e.g.*, (1) $\sin(270°+\theta)$: The angle lies in the fourth quadrant and hence its sign shall be −ve $\therefore$ its vulue $=-\cos\theta$; (2) $\sec(270°-\theta)$: It lies in the 3rd qudrant $\therefore$ its sign is negative and hence its value $=\operatorname{cosec}\theta$; (3) $\cot(90°+\theta)$, it lies in the second quadrant and hence its sign is negative. $\therefore$ $\cot(90°+\theta)$ is $-\tan\theta$.

SOLVED EXAMPLES

1. *Evaluate $\sin 135°$, $\cos 240°$, $\sec 1980°$ and $\cot(-960°)$.*

(*i*) $\sin(135°)=\sin(90°+45°)$

$=\cos 45°$ $\quad[\because \sin\overline{90+\theta}=\cos\theta]$

$=\dfrac{1}{\sqrt{2}}$

(*ii*) $\cos 240° = \cos (180° + 60°)$

$= -\cos 60°$ $\quad [\because \cos (180+\theta) = -\cos \theta]$

$= -\frac{1}{2}$

(*iii*) $\sec (1980°) = \sec (5 \times 360° + 180°)$

$= \sec 180°$ $\quad [\because \sec (n.\ 2\pi + a) = \sec a]$

(*iv*) $\cot (-960°) = -\cot (960°)$ $\quad [\because \cot (-\theta) = -\cot a]$

$= -\cot (2 \times 360° + 240°)$

$= -\cot 240°$ $\quad [\because \cot (n + 2\pi + d) = \cot a]$

$= -\cot (180° + 60°)$

$= -\cot 60°$ $\quad [\because \cot (180 + a) = \cot a]$

$= -\frac{1}{\sqrt{3}}$

2. *Find the value of cot 510°, sec 495° and cosec* $\frac{5\pi}{6}$ (*P.U.*)

(*i*) $\cot 510° = \cot (360° + 150°)$

$= \cot 150° = \cot (180° - 30°)$

$= -\cot 30° = -\sqrt{3}$ $\quad [\because \cot \overline{180° - a} = -\cot a]$

(*ii*) $\sec 495° = \sec (360° + 135°)$

$= \sec 135°$ $\quad [\because \sec (360 + a) = \sec a]$

$= \sec (90 + 45°)$

$= -\text{cosec } 45°$ $\quad [\because \sec (90 + a) = \text{cosec } a]$

$= -\sqrt{2}$

(*iii*) $\text{cosec} \frac{5\pi}{6} = \text{cosec} \left(\pi - \frac{\pi}{6} \right)$

$= \text{cosec} \frac{\pi}{6}$ $\quad [\because \text{cosec } (180 - a) = \text{cosec } a]$

$= 2$

Example 3. *Prove that*

sin 780°. sin 210 + cos 120°. cos 420° = −1

L.H.S. $= \sin (720 + 60) \cdot \sin (180° + 60)$
$+ \cos (180 - 60) . \cos (360 + 60°)$

$= \sin 60° \times [-\sin 60°] + [-\cos 60° \times \cos 60$

$= -\frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} + [-\frac{1}{2} \times \frac{1}{2}]$

$= -\frac{3}{4} - \frac{1}{4} = -1.$

Example 4. *Reduce to its lowest form.*

$$\frac{\cos (90° + \theta) . \sec (-\theta) . \tan (90° - \theta)}{\sec (360° + \theta) . \sin (180° - \theta) . \cot (90° - \theta)}$$ (*P.U. 1944-S*)

$$\text{Expression} = \frac{-\sin\theta \cdot \sec\theta \cdot [-\tan\theta]}{\sec\theta \times -\sin\theta \cdot \tan\theta} = -1.$$

Example 5. *Prove that*

$\cos 24^\circ + \cos 55^\circ + \cos 125^\circ + \cos 204^\circ + \cos 300^\circ = \frac{1}{2}$. (P.U.)

L.H.S. $= \cos 24^\circ + \cos 55^\circ + \cos 125^\circ + \cos 204^\circ + \cos 300^\circ$

$= \cos 24^\circ + \cos 55^\circ + \cos (180^\circ - 55) + \cos (180 + 24^\circ) + \cos (360 - 60^\circ)$

$= \cos 24^\circ + \cos 55^\circ - \cos 55^\circ - \cos 24^\circ + \cos 60^\circ$

$= \cos 60^\circ = \frac{1}{2}$

EXERCISE IV (A)

1. Evaluate $\sin 480^\circ$, $\tan(-945^\circ)$, $\sec 225^\circ$ (P.U.)
2. Find value of $\sin 4620^\circ$, $\cos 225^\circ$, $\tan(-585^\circ)$ (P.U.)
3. Show that

$$\frac{\cos(720^\circ + A) \cdot \cot(180^\circ - A) \cdot \operatorname{cosec}(180^\circ + A)}{\sin(270^\circ + A) \cdot \tan(270^\circ - A) \cdot \sec(90^\circ + A)} = 1.$$

4. Show that $\dfrac{\cos(90^\circ - A) \cdot \cos(180^\circ - A) \cdot \tan(180^\circ + A)}{\sin(90^\circ + A) \cdot \sin(180^\circ - A) \cdot \tan(180^\circ - A)}$

$= 1.$

5. Show that (*i*) $\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8} = 2$

(*ii*) $\sin^2 22\frac{1}{2} + \sin^2 67\frac{1}{2} + \sin^2 112\frac{1}{2} + \sin^2 157\frac{1}{2} = 2.$

[Sol. (*i*) $\cos^2\frac{5\pi}{8} = \cos^2\left(\frac{\pi}{2} + \frac{\pi}{8}\right) = \sin^2\frac{\pi}{8}$

and $\cos^2\frac{7\pi}{8} = \cos^2\left(\frac{\pi}{2} + \frac{3\pi}{8}\right) = \sin^2\frac{3\pi}{8}$

$\therefore$ L.H.S. $= \left(\cos^2\frac{\pi}{8} + \cos^2\frac{5\pi}{8}\right) + \left[\cos^2\left(\frac{3\pi}{8}\right) + \cos^2\left(\frac{7\pi}{8}\right)\right]$

$= \left(\cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8}\right) + \left[\cos^2\frac{3\pi}{8} + \sin^2\frac{3\pi}{8}\right]$

$= 1 + 1 = 2.$

6. Prove that

(*i*) $\tan(2\pi - \theta) - \tan(\pi - \theta) = \cot\left(\frac{\pi}{2} + \theta\right) = \cot\left(\frac{3\pi}{2} + \theta\right)$

(*ii*) $\sin 780^\circ \cos 330^\circ + \cos(-300)\sin(-210^\circ) = 1$

(*iii*) $\sin\left(\frac{\pi}{4}+\theta\right)=\cos\left(\frac{\pi}{4}-\theta\right)$

7. Find all the angles between 0° and 360° which satisfy the equation :—

(*i*) $\sin\theta=\frac{1}{2}$. (*ii*) $\tan\theta=1$. (*iii*) $\sec\theta=-2$.

Sol. (*i*) $\sin\theta=\frac{1}{2}=\sin 30°=\sin(180-30°)=\sin 150°$.

$\therefore\quad \theta=30°$ or $150°$.

8. Find for all +ve values of θ less than 2π, which will satisfy the equations.

(*i*) $\tan\theta+1=0$, (*ii*) $\cos^2\theta=\frac{1}{4}$, (*iii*) $2\cos\theta=3\tan\theta$.

(*P.U.* 1937)

MISCELLANEOUS EXERCISES ON CHAPTER IV

1. For all values of x, prove that

$\tan x.\tan\left(\frac{\pi}{2}+x\right)+1=0$ (*P.U.* 1930)

2. Given $\cot A=\tan(n-1)A$, find one value of A. Find the values of cos :65°, tan 945° sec 1305° and tan 570°.

(*P.U.* 1916)

[**Hint** : (*i*) $\cot A=\tan\left(\frac{\pi}{2}-A\right)$

$\therefore\ \tan\left(\frac{\pi}{2}-A\right)=\tan(n-1)A$

$\therefore\ \frac{\pi}{2}-A=(n-1)A=nA-A\quad\therefore\ \frac{\pi}{2}=nA$]

3. (*a*) Find 'x' from the equation

$\operatorname{cosec}(90°+A)+x\cos A\cot(90°+A)=\sin(90+A)$

(*P.U.* 1936 S)

[**Hint** : The equation can be written as

$\sec A+x.\cos A(-\tan A)=\cos A$

or $\sec A-\cos A=x.\cos A.\tan A$

$\therefore\ \dfrac{\sec A-\cos A}{\cos A\tan A}=x$

Simplify L.H.S.]

(*b*) Simplify $\dfrac{\cos(90+\theta)\sec(-\theta).\tan(180-\theta)}{\sec(360+\theta)\sin(180+\theta).\cot(90-\theta)}$

(*P.U.* 1944-S)

4. Show that each value of any circular function of an angle x is, in general, repeated twice from 0 to 2π. (*P.U.*)

[**Hint** : $\sin(x)=\sin(\pi-x)$ and also $\text{cosec } x=\text{cosec }(\pi-x)$.

$\therefore$ the angles x and $\pi-x$ have the same sine and the same cosecant.

$$\tan x=\tan(\pi+x)\ ;\ \cos x=\cos(2\pi-x)]$$

5. Point out which of the circular functions of x remain unaltered when x is changed into

(*i*) $\frac{\pi}{2}+x$. (*ii*) $\pi-x$. (*iii*) $\pi+x$. (*iv*) $-x$.

(*P.U. 1932*)

6. If A, B, C, D be the angles of a cyclic quadrilateral, prove that $\cos A+\cos B+\cos C+\cos D=0$ (*C.U.*)

[**Hint** : $A+C=180$

$\therefore\ C=180-A$

or $\cos C=\cos(180-A)=-\cos A$

Similarly $\cos C=-\cos B$]

7. If A, B, C are the angles of a triangle, prove that

(*i*) $\sin(A+B)=\sin C$ and $\sin\left(\frac{A+B}{2}\right)=\cos\frac{C}{2}$

(*ii*) $\cos(A+B)=-\cos C$ and $\cos\left(\frac{A+B}{2}\right)=\sin\frac{C}{2}$

(*iii*) $\tan(A+B)=-\tan C$ and $\tan\left(\frac{A+B}{2}\right)=\cot\frac{C}{2}$

8. Prove that (*i*) $\sin(n\pi+\theta)=(-1)^n\sin\theta$

(*ii*) $\cos(n\pi+\theta)=(-1)^n\cos\theta$

(*iii*) $\tan(n\pi+\theta)=\tan\theta$.

[**Hint** : (*i*) $\sin(1.\pi+\theta)=-\sin\theta=(-1)^1\sin\theta$

$\sin(2.\pi+\theta)=+\sin\theta=(-1)^2\sin\theta$

$\sin(3.\pi+\theta)=-\sin\theta=(-1)^3\sin\theta$

$\sin(4.\pi+\theta)=+\sin\theta=(-1)^4\sin\theta$

....

$\therefore\ \sin(n\pi+\theta)=(-1)^n\sin\theta]$.

9. Show that the trigonometrical ratios of any angle can be expressed in terms of trigonometrical ratios of an angle not greater than 45°. (*P.U.*)

Let the angle $=\theta$ (degrees). Divide θ by 90°. If x is the quotient and α is the remainder then

$\theta = x \times 90 + \alpha$; α is either $< 45^\circ$ or $> 45^\circ$.

if α is greater than 45°, put it $= 90 - \alpha'$, where α' is less than 45° $\therefore$ In all cases.

$\theta = n \times 90 + \alpha$ or $= (n+1)90^\circ - \alpha'$

$\therefore$ an $\angle\theta$ can always be expressed as $\theta = k.90 \pm \alpha = k.\frac{\pi}{2} \pm \alpha$, where k is an integer (positive or negative) and α is a positive angle not greater than $\frac{\pi}{4}$.

k may be even or odd.

(a) When k is even, positive or negative, say $2n$.

Then $\sin\theta = \sin\left(2n \times \frac{\pi}{2} \pm \alpha\right)$

$= \sin(n\pi \pm \alpha)$; if n is even and $= 2p$, we get $\sin\theta$

$= \sin(2p\pi \pm \alpha) = \pm\sin\alpha$

If n is odd and $= 2p+1$, we get $\sin\theta$

$= [\sin\overline{(2p+1)}\,\pi \pm \alpha] = \sin(\pi \pm \alpha) = \pm\sin\alpha$

(b) When k is odd, positive or negative, say $2n+1$

$\sin\theta = \sin\left(\overline{2n+1}\,\frac{\pi}{2} \pm \alpha\right) = \sin\left(n\pi + \frac{\pi}{2} \pm \alpha\right)$

$= \sin\left(\frac{\pi}{2} + n\pi \pm \alpha\right)$

$= \cos(n\pi \pm \alpha)$, if n is even and $= 2p$, we have

$\sin\theta = \cos(2p\pi \pm \alpha) = +\cos\alpha$

But if m is odd, say $2p+1$, then we get

$\sin\theta = \cos\overline{(2p+1}\,\pi \pm \alpha)$

$= \cos(2p\pi + \pi \pm \alpha)$

$= \cos(\pi + \alpha) = -\cos\alpha$

Similarly other trigonometric ratios can be expressed in terms of those of α.

ANSWERS TO EXERCISES IN CHAPTER IV

Exercise IV A

1. $\frac{\sqrt{3}}{2}$, -1, $-\sqrt{2}$. 2. $\frac{-\sqrt{3}}{2}$, $-\frac{1}{2}$, -1.

7. (*ii*) 45°, 225°. (*iii*) 120°, 240°.

8. (*i*) 135°, 315°. (*ii*) 60°; 300°; 220°; 240°.

(*iii*) 30°, 150°.

Miscellaneous Exercises on Chapter IV

2. (*i*) $-\frac{1}{\sqrt{2}}$, $-\sqrt{2}$, $\frac{1}{\sqrt{3}}$ 3. (*a*) tan A (*b*) -1.

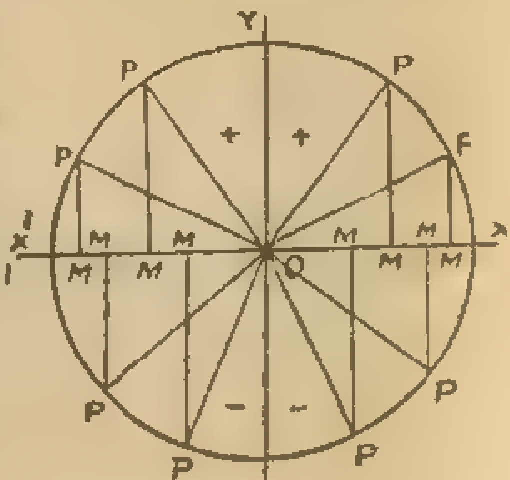
5. (*i*) None, (*ii*) sin x and cosec x, (*iii*) tan x and cot x, (*iv*) cos x and sec x.

CHAPTER X

VARIATION OF TRIGONOMETRICAL RATIOS AND THEIR GRAPHS

5'1. To trace the variations of sin θ as θ increases from 0° to 360°.

Let X'OX and Y'OY be two straight lines intersecting at right angles in O. With centre O and radius equal to unity describe a circle.



Let the revolving line OP starting from OX, trace out $\angle XOP=\theta$ in the anti-clockwise direction. In one complete revolution it will describe all angles from 0° to 360°. Draw PM $\perp$ X'OX fory an position of P. Then $\sin\theta=\dfrac{MP}{OP}=\dfrac{MP}{1}$

So the value of sin θ depends on the chadge in the values of MP as OP is constant and equal to 1.

First Quadrant. As the point P moves from X to Y, angle MOP increases from 0° to 90°.

In this case MP increases from zero to 1 (initially when P had coincided with X its value was zero, then it began to increase and the greatest value equal to 1 was reached when P coincided with Y).

$\therefore$ sin θ has increased from 0 to 1 and remains +ve.

Thus in the first quadrant as θ varies from 0° to 90°, sin θ is postive and varies from 0 to 1, *i.e.*, increases from 0 to 1.

Second Quadrant. As the point P moves from Y to X' the $\angle$MOP increases from 90° to 180°.

PM decreases from 1 to 0.

[$\therefore$ At Y, MP $=$ OY $=1$ and at X, PM $=0$] and remains positive.

$\therefore$ sin θ changes from 1 to 0 and is positive.

Thus in the second quadrant as θ varies from 90° to 180°, sin θ is positive and varies from 1 to 0 or decreases from 1 to 0.

Third Quadrant. Here θ changes from 180° to 270° and MP is negative and increases *numerically* or decreases *algebraically* from 0 to -1 $\therefore$ $\frac{MP}{OP}$ or sin θ is negative and decreases algebraically from $\frac{0}{1}$ to $\frac{-1}{1}$ or from 0 to -1.

Thus in the third quadrant as θ varies from 180° to 270°, sin θ is negative and varies from 0 to -1 or decreases algebraically from 0 to -1.

Fourth Quadrant. Here θ changes from 270° to 360° and MP is still negative and decreases *numerically* or increases algebraically from -1 to 0.

$\therefore$ $\frac{MP}{OP}$ or sin θ is negative and increases algebraically from $\frac{-1}{1}$ to $\frac{0}{1}$ or from -1 to 0.

Thus in the fourth quadrant as θ varies from 270° to 360°, sin θ is negative and varies from -1 to 0 or increases algebraically from -1 to 0.

Note 1. Sin θ is never greater than unity and that it assumes all values between 1 and -1.

Note 2. There are two angles lying between 0° and 360°, which have a given sine. If the given sine is positive, the two angles lie between 0° and 180° and if the given sine is negative, the two angles lie between 180° and 360°.

5·2. To draw the graph of sin θ as θ increases from 0° to 360°.

(*i*) Put $y=\sin x$, then we have

Table of values

x	0°	30°	60°	90°	120°	150°	180°
$y=\sin x$	0	·5	·87	1	·87	·5	0
x	210°	240°	270°	300°	330°	360°	
$y=\sin x$	$-$·5	$-$·87	$-$1	$-$·87	$-$·5	0	

Let OX and OY be the axis of X and Y respectively.

(*ii*) *Scale.* Let 1 small division along the X axis $=10^\circ$
and 1 „ „ „ „ Y-axis $=$·1

(*iii*) Plot the points given by the above table and draw a smooth curve through them and we get the required graph.

Note. (*i*) It is clear that the graph is continuous and therefore the function sin x is continuous.

Note. (*ii*) The graph could also be drawn for negative values of the angle θ. We would find corresponding values of y. The graph would be drawn towards the left of the Y-axis.

5·3. To trace the variations of $\cos\theta$ as θ increases from 0° to 360°.

Draw the Fig. as in Art. 5·1

$$\therefore \quad \cos\theta = \frac{OM}{OP} = \frac{OM}{1} = OM$$

OP being constant equal to unity the variations in $\cos\theta$ depend upon the values of OM.

First Quadrant. $\angle MOP$ or θ changes from 0° to 90°.

When $\theta=0$, M and P coincide at X $\therefore$ OM $=$ OP $=1$ and hence $\cos 0^\circ = OM = 1$.

As θ increases OM decreases and changes from 1 to 0 ($\because$ OM is $=1$ when P coincides with X and is zero when P coincides with Y).

Thus in the first quadrant as θ increases from 0° to 90° $\cos\theta$ is positive and decreases from 1 and 0.

Second Quadrant. Here θ changes from 90° to 180°, OM, which is now on the left of YY′, is negative and increases *numerically* from O to OX′.

$\therefore$ $\dfrac{OM}{OP}$ changes from $\dfrac{0}{1}$ to $\dfrac{-1}{1}$, *i.e.*, $\cos\theta$ changes from 0 to -1 and is negative.

Thus in the second quadrant as θ increases from 90° to 180°, $\cos\theta$ is negative and decreases algebraically or increases numerically from 0 to -1.

Third Quadrant. As the point P moves from X′ to Y′ θ increases from 180° to 270°. OM is still negative and decreases numerically from OX′ to O ($\because$ It is equal to 0 when P coincides with Y′, *i.e.* when $\angle MOP$ or $\theta = 270^\circ$) $\therefore$ $\dfrac{OM}{OP}$ changes from $\dfrac{-1}{1}$ to $\dfrac{0}{1}$, *i.e.*, $\cos\theta$ changes from -1 to 0 and is negative.

Thus in the third quadrant as θ increases from 180° to 270°, $\cos\theta$ is negative and varies from -1 to 0, *i.e.*, decreases numerically or increases algebraically from -1 to 0.

Fourth Quadrant. As θ increases from 270° to 360° OM is positive and increases from 9 to OX ($\because$ It is equal to OX when P coincides with X, *i.e.*, when $\angle$MOP or $\theta=360°$).

$\therefore$ OM changes from 0 to 1, *i.e.*, cos θ changes from 0 to 1 and is positive.

$\therefore$ in this quadrant as θ increases from 270° to 360°, cos θ is positive and increases from 0 to 1.

Note 1. cos θ is never greater than unity and is capable of assuming all values between +1 and −1.

Note 2. There are two angles lying between 0° and 360° which have a given cosine. If the given cosine is positive, one of the angles is between 0° and 90° and the other between 270° and 360° and if the given cosine is negative, then both the two angles lie between 90° and 270°.

5·4. To draw the graph of cos θ as θ increases from 0° to 360°. (*P.U. 1941*)

(*i*) $y=\cos x$. Then we have the

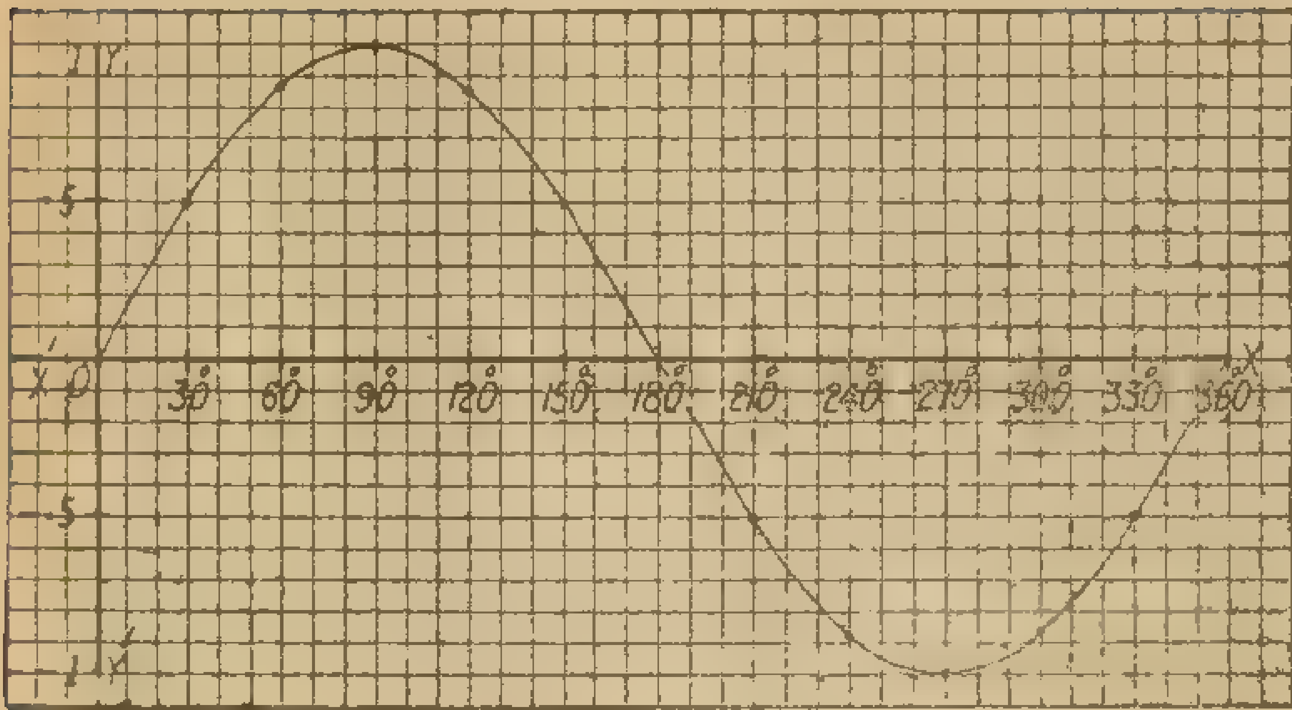
Tables of values

x	0	30°	60°	90°	120°	180°	180°
$y=\cos x$	1	·87	·5	0	−·5	−87	−1
x	210°	240°	270°	300°	330°	3°06	
$y=\cos x$	−·87	−·5	0	·5	87	1	

Let OX and OY be the axes of z and y respectively.

(*ii*) Scale : Let the side of a small square along the X-axis represent 10° and let the side of a small square along the Y-axis represent 1.

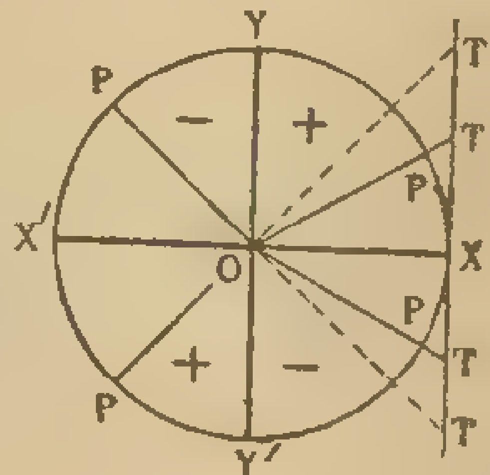
(*iii*) If the points given by the above table are plotted and joined by a free-hand curve, we get the required graph.



Note : The function cos x is also continuous as the graph is continuous.

5·5. To trace the variations of tan θ as θ increases from 0° to 360°.

Let XOX′ and YOY′ be two straight lines intersecting at O. With O as centre and radius equal to unity draw a circle. Let a revolving line OP (=1) starting from OA trace out ∠XOP=θ in counter clockwise direction. In one complete revolution it will describe all angles from 0° to 360°. Draw the tangent at X to meet OP produced in T.



Then from the △OXT.

$$\tan\theta=\frac{XT}{OX}=XT\ (\because\ OX=1)$$

∴ the variations of tan θ are the same as those of XT.

[XT is + ve if it is drawn above OX and − ve if drawn below OX.]

First Quadrant. As θ increases from 0° to 90°, XT falls above X and

$\therefore$ XT is positive and increases from 0 to ∞ [$\therefore$ when $\theta = 0°$ T coincides with X and when P = 90° OP coincides with OY which is parallel to XT, and hence meet AT line at ∞ $\therefore$ AT $\rightarrow \infty$.]

$\therefore$ tan θ is positive and increases from 0 to ∞

First Quadrant. When θ is a little greater than 90°, T falls below X and tan θ is negative but very large $\therefore$ when θ passes through 90°, tan θ suddenly changes from $+\infty$ to $-\infty$.

$\therefore$ As θ increases from 90° to 180.

AT is negative and changes from $-\infty$ to 0.

$\therefore$ tan is negative and increases algebraically from $-\infty$ to 0.

Third Quadrant. As θ increases from 180° to 270°.

AT lies above OX and $\therefore$ is positive and increases from 0 to ∞.

$\therefore$ tan θ is positive and increases from 0 to $+\infty$.

Fourth Quadrant. As θ passes through 270° tan θ suddenly change from $+\infty$ to $-\infty$ (as in the case of 90°).

$\therefore$ as θ increases from 270° to 360°,

AT is negative and changes from $-\infty$ to 0.

$\therefore$ tan θ is negative and increases algebraically from $-\infty$ to 0.

551. To draw the graph of tan θ as θ increases from 0° to 360°.

(*i*) $y = \tan x$, then we have the

Tables of values

x	0	30°	60°	90−0°	90+0°	120°	150°	180°
$y = \tan x$	0	·58	1·7	$+\infty$	$-\infty$	−1·7	−180°	0

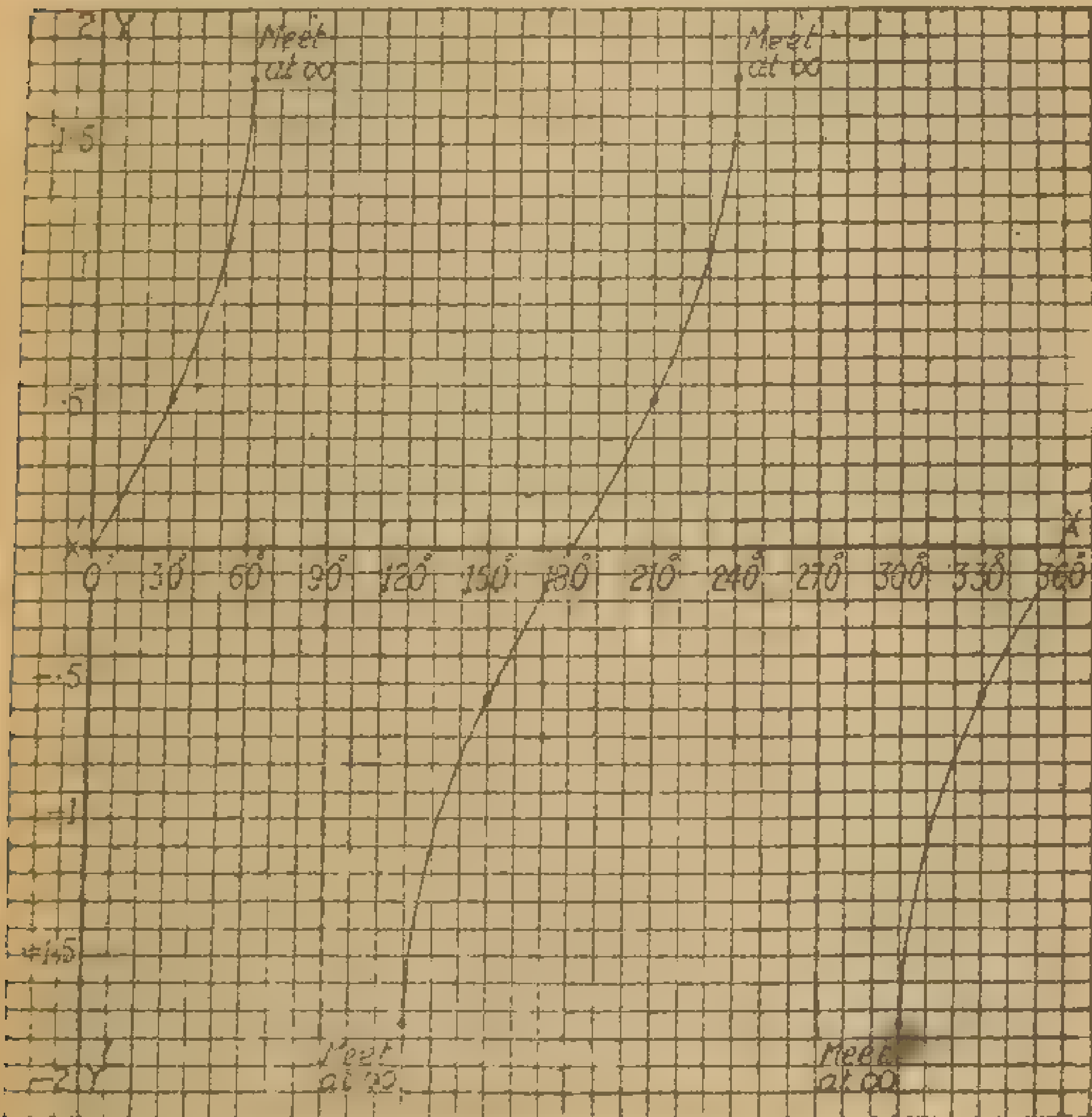
x	210°	240°	270°−0	270°+0	300°	380°	360°
$y = \tan x$	·58	1 7	$+\infty$	$-\infty$	−1·7	−·58	0

Note : $\tan (90-0)^\circ$ or $\tan (270-0^\circ) = +\infty$ means that if the angle x is slightly less than 90° or 270° and $\tan x$ is $+\infty$ and if the angle x is $90^\circ+0^\circ$ or $270^\circ+0^\circ$ or slightly greater than 90° or 270°, $\tan x$ is $-\infty$.

(*ii*) Let OX and OY be the axes of x and y respectively.

Scale :—Let the side of a small square represent 10° along the X-axis and let the side of a small square represent 1 along Y-axis.

(*iii*) If the points given by the above table are plotted and joined by a free hand curve, we get the required graph.



Note. (*i*) The graph has a break when $x=90^\circ$ and 270°; therefore the function $\tan x$ is discontinuous at each of these points.

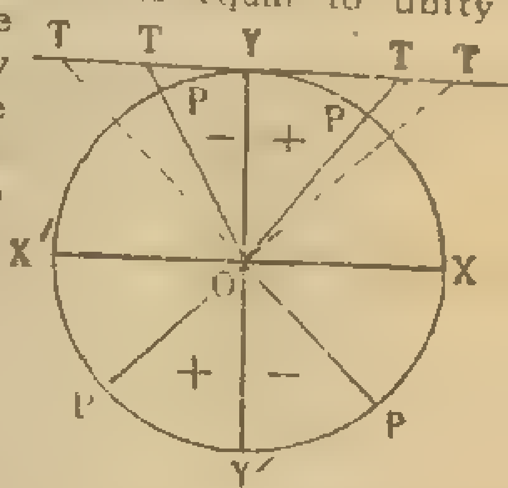
(*ii*) $\tan\theta$ is capable of assuming any real value.

(*iii*) As θ passes through 90°, $\tan\theta$ suddenly changes from $+\infty$ to $-\infty$; similarly $\tan\theta$ changes from $+\infty$ to $-\infty$, when θ passes through 270°.

(*iv*) Corresponding to a given tangent there are two angles between 0° and 360°. If the given tangent is positive, one of the angles lies between 0° and 90° and the other between 180° and 270°, but if the given tangent is negative, then one of the angles lies between 90° and 180° and the other between 270° and 360°.

5·6. Trace the variations of the value of $\cot\theta$ from 0 to 390.

Let XOX' and YOY' be two straight lines intersecting at right angles in O. With centre O and radius equal to unity describe a circle. Let a revolving line OP starting from OX trace out any angle $XOP=\theta$ in counter-clockwise direction. In one complete revolution it will describe all angles from 0° to 360°. Draw the tangent at Y to meet OP produced in T.



Then $\cot\theta=\cot XOP=\cot OTY$

$$=\frac{YT}{OY}=YT.$$

$$(\because OY=1)$$

$\therefore$ The variations of $\cot\theta$ are the same as those of BT. YT is +ve if it is drawn to the right of Y, and −ve if drawn to the left of Y.

First Quadrant. As θ increases from 0° to 90°, YT is positive and decreases from ∞ to 0 [$\because$ when $\theta=0$, OP coincides with OX which is parallel to YT and hence meets BT at infinity and therefore Y is ∞ and when $\theta=90^\circ$, P coincides with Y and $\therefore$ $YT=0$].

$\therefore$ $\cot\theta$ is positive and decreases from ∞ to 0.

Second Quadrant. As θ increases from $90°$ to $180°$, YT is negative and changes from 0 to $-\infty$. [$\because$ when $\theta=90°$, YT$=0$ and when $\theta=180°$ OP coincides with OX which is parallel to YT and hence meets YT at infinity and $\therefore$ YT$=0 - \infty$].

$\therefore$ cot θ is negative and decreases algebraically from 0 to $-\infty$.

Note :—When θ is slightly less than $180°$, YT is negative and very large, when θ is slightly greater than $180°$, tan θ suddenly changes from $-\infty$ to $+\infty$.

Third Quadrant As θ increases from $180°$ to $270°$, YT is positive and decreases from ∞ to 0 [$\because$ when θ is slightly greater than $180°$, OP meets YT at infinity as it coincides with OX′ and hence parallel to YT when $\theta=270°$. P coincides with Y′ $\therefore$ YT$=0$].

$\therefore$ cot θ is positive and decreases from ∞ to 0.

Fourth Quadrant. As θ increases from $270°$ to $360°$, YT is negative and changes from 0 to $-\infty$.

$\therefore$ cot θ is negative and decreases algebraically from 0 to $-\infty$.

As θ passes through $360°$, cot θ suddenly changes from $-\infty$ to $+\infty$.

5·61. To draw the graph of cot θ as θ increases from $0°$ to $360°$.

(1) Put $y=\cot x$: then we have the

Table of values

x	$0°$	$30°$	$60°$	$90°$	$120°$	$150°$	$180-0$	$180+0$
$y=\cot$	∞	1·	58	0	$-$·58	$-1·7$	$-\infty$	$+\infty$

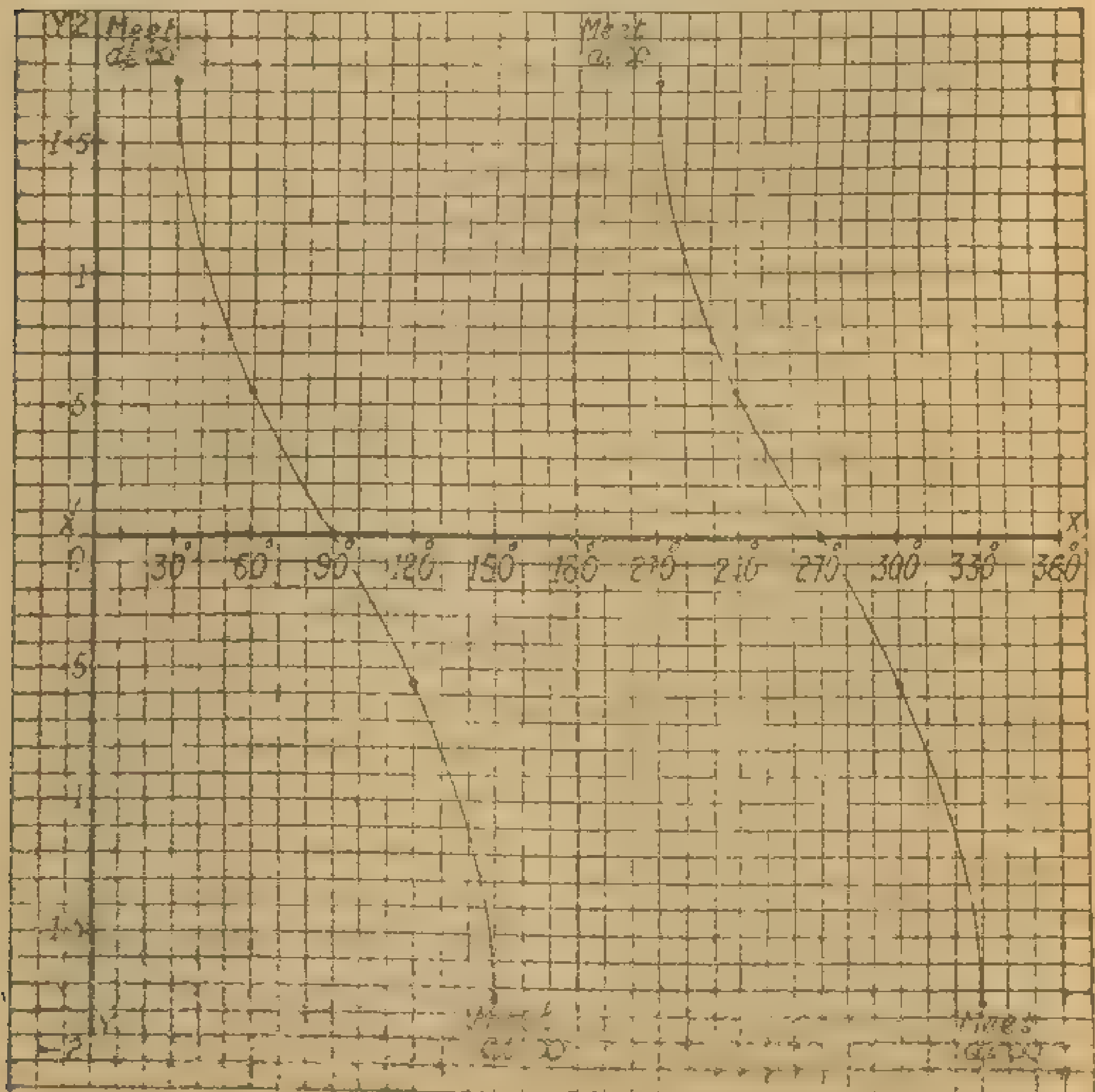
x	$210°$	$240°$	$270°$	$300°$	$330°$	$360°-0$	$360°+0$
$y=\cot$	1·7	·58	0	$-$58	$-1·7$	$-\infty$	$+\infty$

Cot $180^\circ - 0$ or $360^\circ - 0 = \infty$ means that if the angle x is slightly less than 180° or 360°, cot x is $-\infty$; and cot $(180^\circ + 0)$ and cot $(360 + 0) = \infty$ means that the angle x is slightly greater than 180° or 360° or cot x is $+\infty$.

(*i*) Let OX and OY be the axes of x and y respectively.

Scale. Let the side of a small square represent 10° along X-axis and let the side of a small square represent ·1 along Y-axis.

(*ii*) If the points given by the above table are plotted and joined by a freehand curve, we get the required graph.



Properties of cot θ

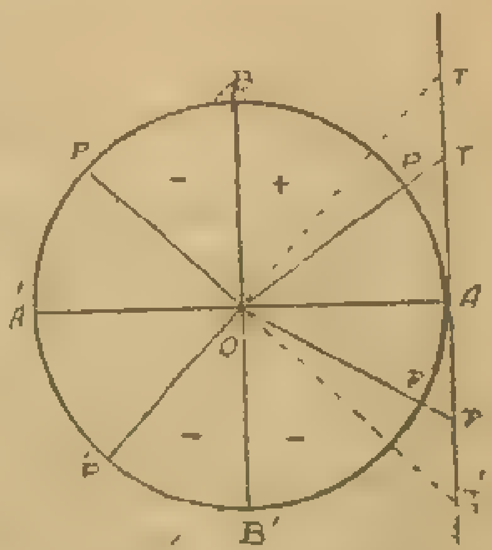
Note 1. Cot θ is capable of assuming all real values.

Note 2. As θ passes through $180°$ or $360°$, cot θ changes suddenly from $-\infty$ to $+\infty$.

Note 3. Corresponding to a given cotangent there are two angles between 0 to 360°. If the given cotangent is positive, one of the angles lies between 0° and 90° and the other between 180° and 270°; but if the given cotagent is negative, then one of the angles lies between 90° and 180° and the other between 170° and 360°.

Note 4. As the graph has a break when $x=180°$ or $360°$, therefore the function cot x is discontinuous at each of these points.

57. To trace the variations of sec θ as θ increases from 0° to 360°.



Let A'OA and BOB' be two st. lines meeting at rt ∠s in O, with centre O and radius $=1$, draw a circle. Let the revolving line, starting from OA, trace out an angle θ, revolving through the four quadrants. Draw the tangent at A to meet OP produced in T.

Then from $\triangle$OAT;

$$\sec\theta=\frac{OT}{OA}=\frac{OT}{1}=OT. \quad [\because OA=1].$$

$\therefore$ **OT represents sec AOP.**

Sec θ is positive if OT is drawn along OP produced bounding the angle θ; but if OT lies along OP produced backwards or on the opposite side of O, sec θ is negative, thus in figure sec AOP' is negative since it is represented by OT' on the opposite side of O to OP'.

First Quadrant. As θ increases from 0° to 90°, OT is positive and increases from 1 to ∞ [$\because$ when $\theta=0$, OT coincides with OA$=1$ and when $\theta=90°$, OP coincides with OB which is parallel of AT and hence OT is infinite].

$\therefore$ sec θ is positive and increases from 1 to ∞.

When θ is slightly less than $90°$, OT is infinitely large and positive.

When θ is slightly greater than $90°$ OP′ meets the tangent at A, in T′ when produced backward and hence OT′ or OT becomes negative and is infinitely large and positive.

$\therefore$ When θ passes through $90°$, sec θ suddenly changes from $+\infty$ to $-\infty$.

Second Quadrant. As θ increases from $90°$ to $180°$,

OT is negative and changes from $-\infty$ to -1.

[$\because$ when θ is slightly $>90°$, OT is $-\infty$ and when $\theta = 180°$ $OT' = OA = -1$].

$\therefore$ sec θ is negative and increases algebraically from $-\infty$ to -1.

Third Quadrant. As θ increases from $180°$ to $270°$, OT is negative and changes from -1 to $-\infty$.

$\therefore$ sec θ is negative and decreases algebraically from -1 to $-\infty$.

When θ passes through $270°$, sec θ suddenly changes from $+\infty$ to $-\infty$.

Fourth Quadrant. As θ increases from $270°$ to $360°$. OT is positive and decreases from $+\infty$ to 1.

$\therefore$ sec θ is positive and decreases from $+\infty$ to 1.

Note 1. Sec θ never lies between 1 and -1 and is capable of assuming all other real values.

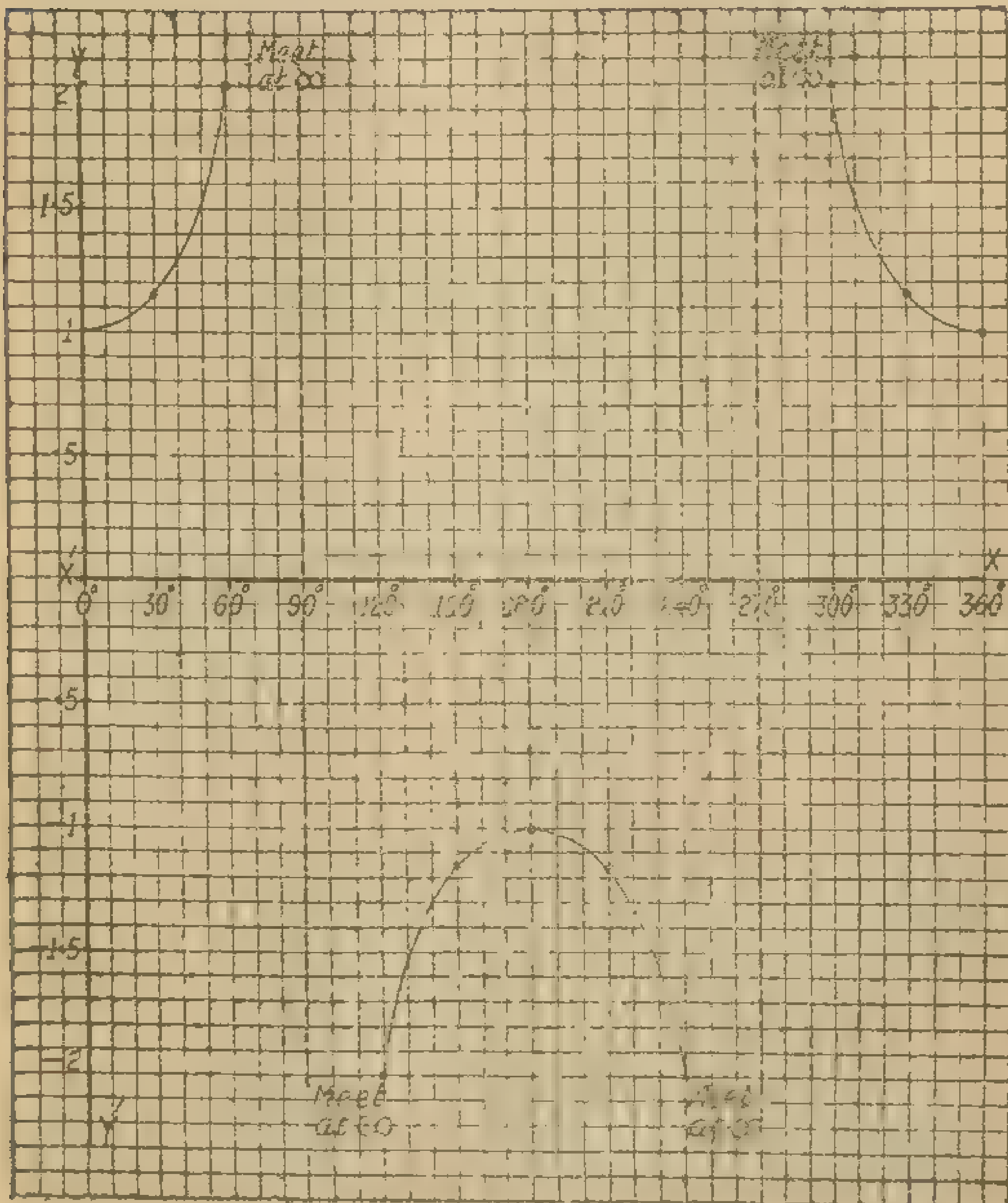
Note 2. It should be noted that when a trigonometric ratio becomes infinite, it must change sign from $+\infty$ to $-\infty$ or from $-\infty$ to $+\infty$; in the case of sec θ, we find that it changes from $+\infty$ to $-\infty$ at $\theta = 90°$ and from $-\infty$ to $+\infty$ at $\theta = 270°$.

Note 3. There are two angles between $0°$ and $360°$ corresponding to a given secant. If the given secant is positive, one of the angles lies between $0°$ and $90°$ and the other between $270°$ and $360°$; on the other hand if secant θ is negative, the angles lie between $90°$ and $180°$ and $270°$.

5·71. To draw the graph of sec x as x increases from $0°$ to $360°$.

(*i*) $y = \sec x$, then we have the

x	0°	30°	60°	90° − 0°	90° + 0°	120°	150°	180°
$y = \sec x$	1	1·2	2	$+\infty$	$-\infty$	−2	−1·2	−1
x	210°	240°	270° − 0°	270° + 0°	330°	330°	360°	
$y = \sec x$	−1·2	−1·2	$-\infty$	$+\infty$	+2	1·2	360°	



Sec $(90°-\theta)=+\infty$, means that when x is slightly less then 90°, sec $=+\infty$; when x is slightly greater than 90°, we write sec $(90°+0°)=-\infty$.

Similarly sec $(270°-0°)=-\infty$ means that when θ is slightly less than 270°, sec $\theta=-\infty$ and sec $(270°+0°)=+\infty$ means that when θ is slightly greater than 270°, sec $\theta=+\infty$.

(*ii*) Let OX and OY be the axes.

Scale: Let the side of small square represent 10° along X-axis.

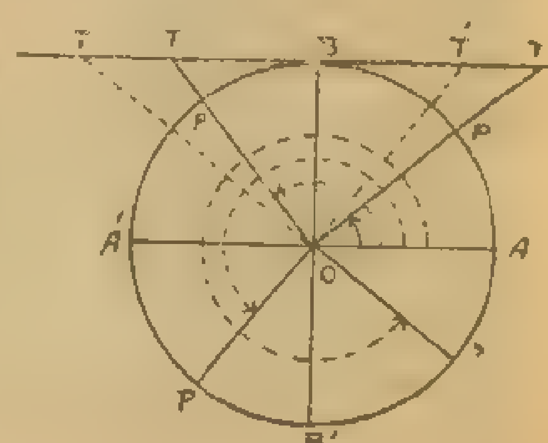
and let the side of a small square represent 1 along Y-axis.

(*iii*) If the points given by the above table are plotted and joined by a free hand curve we get the required graph.

Note. As the graph has a break when $x=90°$ or 270°, the function sec x is discontinuous at each of these points.

58. To trace the variations of cosec θ as θ increase from 0° to 360°.

Let the revolving line OP $(=1)$; starting from OA trace out an angle AOP $=\theta$, revolving through the four quadrants. Draw the tangent at B to meet OP produced in T or T.



Now cosec θ = cosec AOP

= cosec OTB

[$\because$ BT $\parallel$ OA].

$=\dfrac{OT}{OB}=OT$

[$\because$ OB $=1$.]

$\therefore$ Variations of cosec θ are the same as those of OT.

Cosec $\theta=$OT is positive if OT lies along OP produced in the OP direction through O and it is negative if the line OP of the positive angle meets the tangent at B at a point T′ in OP produced backwards in the direction of PO.

First Quadrant. As θ increases from 0° to 90°, OT is positive and decreases from ∞ to 1

[$\because$ When $\theta=0°$, OP coincides with OA which is parallel to BT and hence OT is infinite, when $\theta=90°$, P coincides with B, OT=OB=1].

$\therefore$ cosec θ is positive and decreases from ∞ to 1.

Second Quadrant. As θ increases from 90° to 180°.

OT is positive and increases from 1 to ∞

$\therefore$ cosec is θ positive and increases from 1 to ∞.

When θ is slightly less than 180°, OT is positive, and very large ;

When θ is slightly greater than 180°, OT (drawn along OP produced backwards) is negative and very large ;

$\therefore$ as θ passes through 180°, cosec θ suddenly changes from $+\infty$ to $-\infty$.

Third Quadrant As θ increases from 180° to 270°.

OT is negative and changes from $-\infty$ to -1.

$\therefore$ cosec θ is negative and decreases algebraically from $-\infty$ to -1.

Fourth Quadrant. As θ increases from 270° to 360°

OT is negative and changes from -1 to $-\infty$.

$\therefore$ cosec θ is negative and decreases algebraically from -1 to $-\infty$.

As θ passes through 360°, cosec θ suddenly changes from $-\infty$ to $+\infty$.

Note 1. Cosec θ never lies between $+1$ and -1 and is capable of assuming all other real values.

Note 2. As θ passes through 180°, cosec θ suddenly changes from $-\infty$ to $+\infty$.

Note 3. Corresponding to a given cosecant, there are two angles between 0° and 360°. If the given cosecant is $+$, the angles lie between 0° and 180°; it the given cosecent is negative, the angles lie between 180° and 360°.

5 81. To draw the graph of cosec x as x increases from 0°, to 360°.

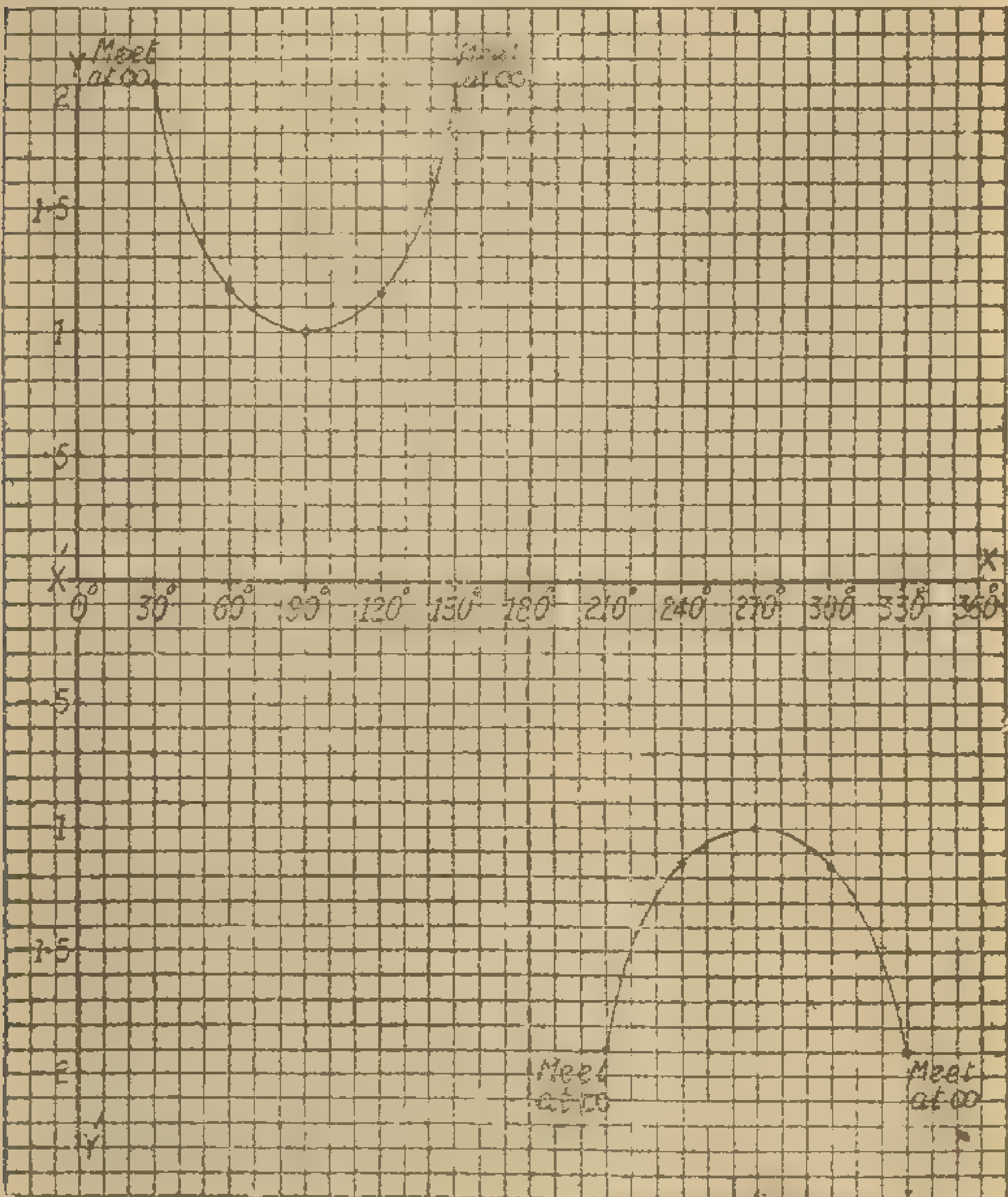
Put $y =$ cosec x, then we have the

Table of values :

x	-0	$+0^\circ$	30°	60°	90°	120°	150°	$180^\circ-0^\circ$	$180^\circ+0^\circ$
$y=$ cosec x	$-\infty$	$+\infty$	2	1·2	1	1·2	2	$+\infty$	$-\infty$

x	210°	240°	270	300°	330°	$363^\circ -0^\circ$	$360^\circ+0^\circ$
$x=$ cosec x	-2	$-1\ 2$	-1	$-1·2$	-2	$-\infty$	$+\infty$

Cosec $180-0^\circ=\pm\infty$ means that if the angle x is slightly less than 180°, cosec $x=\pm\infty$ and cosec $(180^\circ+0^\circ)$ means that if x is slightly greater than 180°, cosec x is $-\infty$. Similarly cosec $360^\circ-0^\circ=-\infty$ means that the angle x is silightly iess than 360°, cosec x is $-\infty$ and cosec $(360+\theta)=+\infty$ means that if x is slightly greeter then 360°, cosec x is $+\infty$.



(*ii*) Let OX and OY be the axis of x and y respectively.

Scale : Let the side of a small square represent $10°$ along x-axis

and let the side of a small square represent ·1 along y-axis.

(*iii*) If the points given by the above Table are plotted and joined by a freehand curve, we get the required graph.

Note. The graph has a break when $x=180°$ or $360°$, therefore the function cosec x is discontinuous at each of these points.

Example 1. *Draw the graph of* $\cos x$ *from* $x=-\pi$ *to* $x=\pi$.

(*i*) Put $y=\cos x$, then we have the (*P.U.*)

Table of values

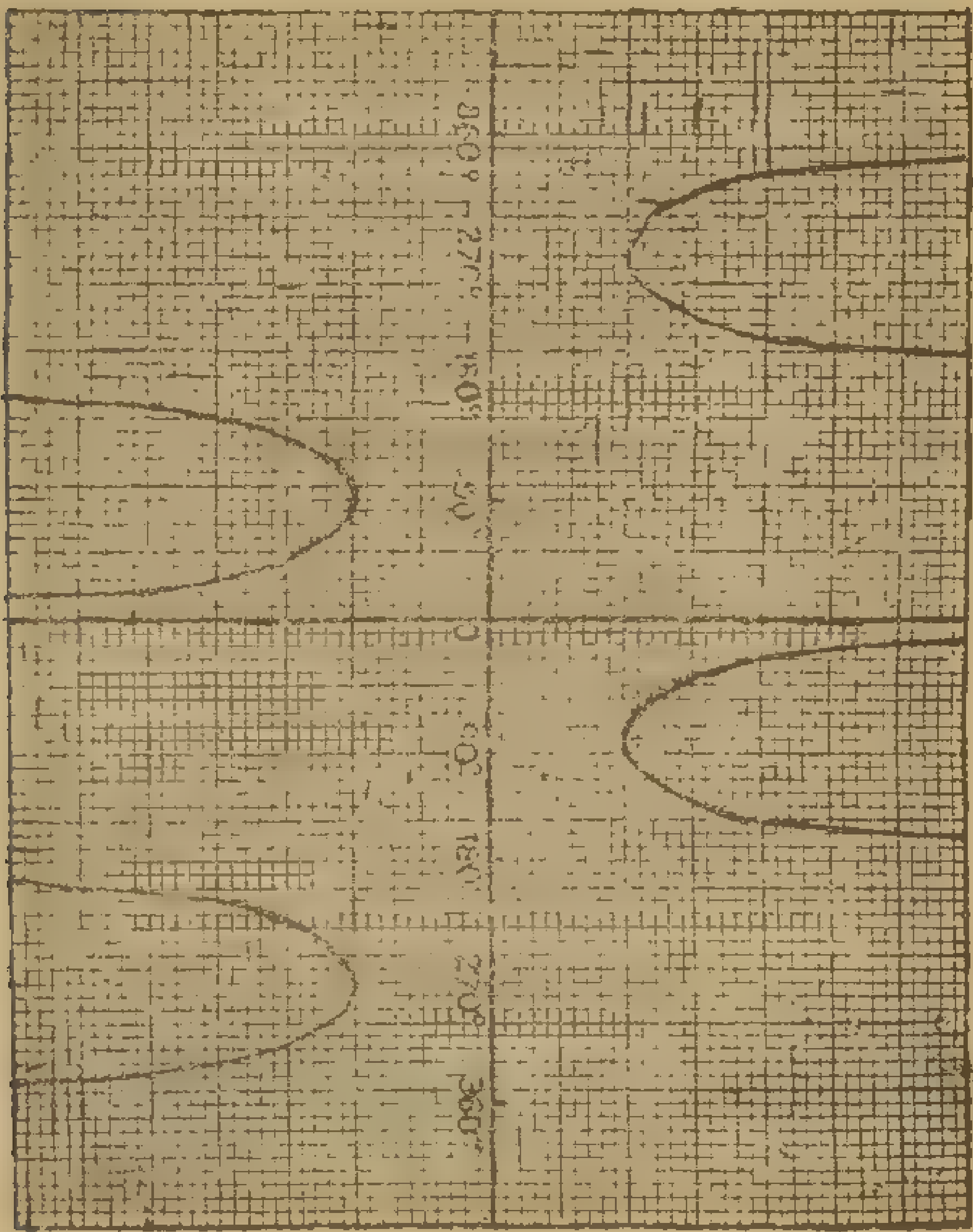
x	$-180°$	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$0°$
$y=\cos x$	-1	$-·87$	$-·5$	0	$·5$	$·87$	1
x	$30°$	$60°$	$90°$	$120°$	$150°$	$180°$	
$y=\cos x$	$·87$	$·5$	0	$-·5$	$-·87$	1	

(*ii*) Let XOX′ and YOY′ be the axis of x and y respectively, the origin O being in the middle of the page.

Positive angles are to be represented along the positive side of the x-axis, *i.e.*, to the right side of O ; and the negative angles should be represented along negative side of the x-axis *i.e.*, towards the left of O.

Scale. Let the side of a small square represent $10°$ along x-axis and let the side of a small square represent ·1 along y-axis.

(*iii*) Plot the points given by the above Table. Join them by a freehand curve and we get the required graph.


360°
270°
180°
90°
0
90°
180°
270°
360°

Example 2. *Draw the graph of cosec x between the limits -2π and $+2\pi$.*

(*i*) Put $y=\text{cosec } x$, then we have the

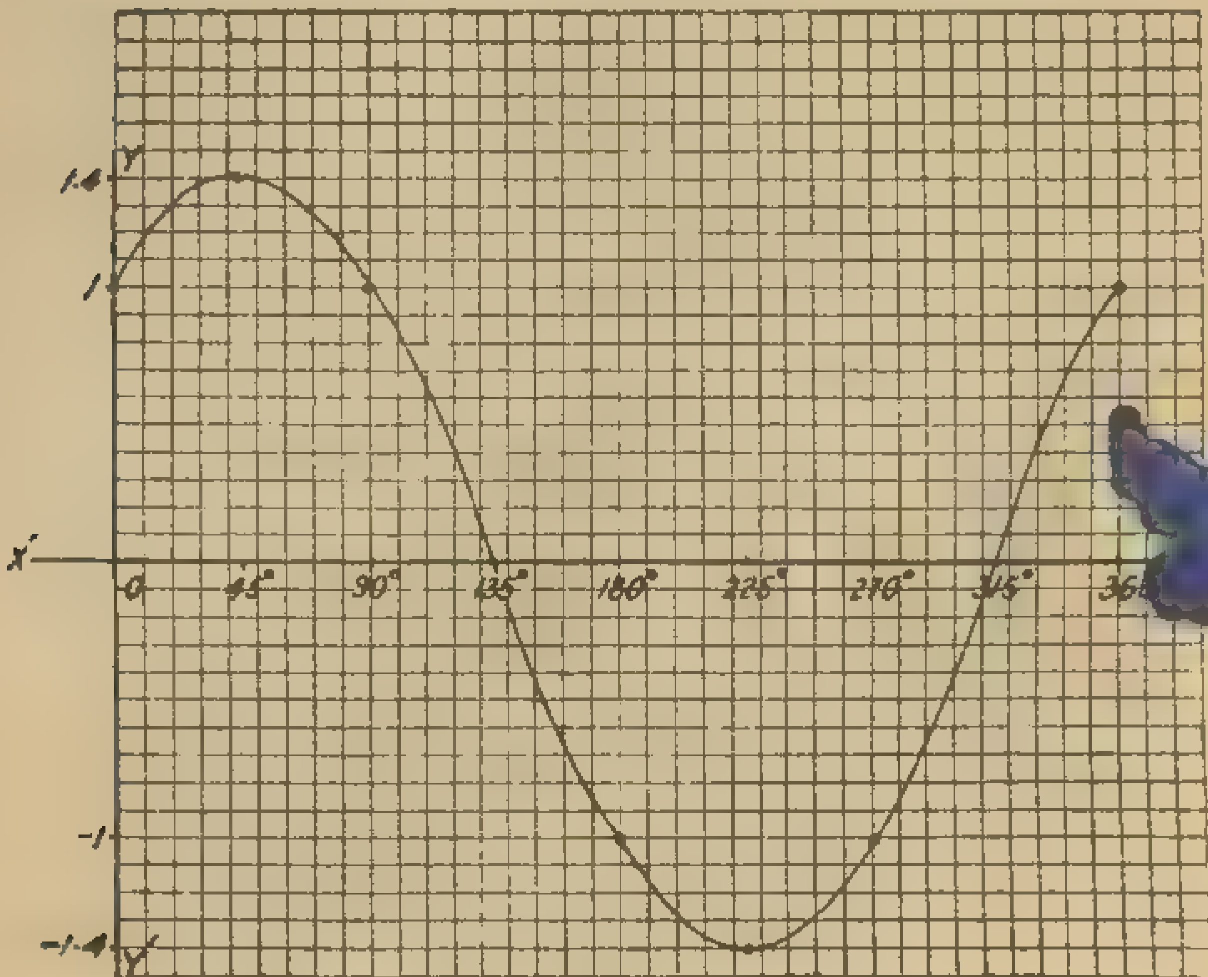
Table of values.

x	$-360°$	$-330°$	$-330°$	$-270°$	$-240°$	$-210°$	$-180°-0°$	$180°+0$
$y=\text{cosec}$	$+\infty$	2	1·2	1	1·2	2	$+\infty$	$-\infty$
x	$-150°$	$-120°$	$-90°$	$-60°$	$-30°$	$0-$	$+0°$	$30°$
$y=\text{cosec}$	-2	$-1·2$	-1	$-1·2$	-2	$-\infty$	$+\infty$	2
x	$60°$	$90°$	$120°$	$150°$	$180°-0°$	$180°+0°$	$210°$	
$y=\text{cosec}$	1·2	1	1.2	2	$+\infty$	$-\infty$	-2	
	$240°$	$270°$	$300°$	$330°$	$360°-0°$	$360°+0$		
$y=\text{cosec}$	$-1·2$	-1	$-1·2$	-2	∞	$+\infty$		

(*ii*) Let XOX′ and YOY′ be the axes drawn in the middle of the sheet.

Scale. Let the side of a small square represent 10° along X-axis and let the side of a small square represent ·1 along Y-axis. The positive angles are to be represented along the positive side of the X-axis; and the negative angles along the negative side, *i.e.*, towards the left of the origin O.

(*iii*) Plot the points given by the above Table. Join them by a freehand curve and we get the required graph. Actual graph is left to the student as an exercise.



Example 3. *Draw the graph of sin x + cos x as x varies from 0° to 360°.*

Let $y = \sin x + \cos x$ and make a chart as shown.

x	0	30	45	60	90	120	135	150	180	210
sin x	0	·5	·71	·87	1	·87	·71	·5	0	−·5
cos x	1	·87	·71	5	0	−·5	−.71	−·87	−1	−·87
y = cos + sin x	1	1·87	1·42	1·87	1	·87	0	−·87	−1	−1·87

x	225	240	270	300	315	330	360
$\sin x$	$-\cdot71$	$-\cdot87$	-1	$-\cdot87$	$-\cdot71$	$-\cdot5$	0
$\cos x$	$-\cdot71$	$-\cdot5$	0	$\cdot5$	$\cdot71$	$\cdot87$	1
$y=\cos+\sin x$	$-1\cdot42$	$-1\cdot37$	-1	$-\cdot37$	$\cdot0$	$\cdot37$	1

$\frac{1''}{10}$ along x-axis represent $10°$ and $1'$ along y-axis [illegible]nt 1. Plot the points given by the above chart and draw a [illegible]te curve through them, we get the above.

Example 4. *Draw the graph of $y=\sin x$ as varies from $0°$ and $180°$ and from the graph find out the values of x when* (*i*) $\sin x=3$, (*ii*) $\sin x=6$. Also find $\sin 65°$.

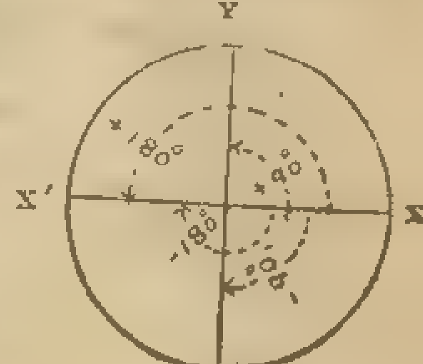
Refer to graph of Art. 26. Read the abscissæ of the points where $y={}^{\cdot}3$ cuts the graph. The angle is about $17°$ or $163°$. Again read the abscissæ of the points where $y={}^{\cdot}6$ meets the graph. The angle s about $37°$ or $143°$.

Take a point M on X-axis so that OM represent $65°$ (*i.e.* $OM=\frac{65°}{11}=6\cdot5$ small squares). At M draw a $\perp$ MP to meet the graph in P. Then $MP=9$ small divisions $=9\times{}^{\cdot}1={}^{\cdot}9$ $\therefore \sin 65°={}^{\cdot}9$.

EXERCISE V (A)

1. Trace the variations of $\sin\theta$ as θ increases from $-180°$ to $+180°$. (*P.U. 1942-S*)

[**Hint** : The initial value of $\theta=(-180°$ to $-90°)$ means that θ lies in the third quadrant ; θ from $-90°$ to $0°$ lies in the fourth quadrant ; θ from $0°$ to $90°$ lies in the first quadrant and θ from $90°$ to $180°$ lies in the second quadrant.]



2. Trace the changes in the values of $\cos x$ between $-\pi$ to $+\pi$ and draw the graph between these limits. (*P.U. 1940-43*)

3. Trace the changes in the values of secant θ between -360° to $+360^\circ$ and draw the graph between these limits.

(*P.U. 1944*)

[**Hint** : Take the initial value of $\theta = -360^\circ$, then θ from -360° to -270° lies in the first quadrant ; θ from -270° to -180° lies in the second quadrant ; θ from -180° to -90° lies in the third quadrant ; θ from -90° to 0 lies in the fourth quadrant ; changes for θ from 0 to 360° have already been discussed.]

4. Trace the changes in the values of cosec of positive and negative values of θ and draw the complete graph. State the period of cosec θ (*P.U.* [illegible])

[**Hint** : θ varies from -360° to $+360^\circ$.]

5. Solve graphically the equations

(*i*) (*a*) $\sin x = \frac{4}{5}$
(*b*) $\sin x = \frac{3}{5}$ } between $-\pi$ to $+\pi$ (P.U. 1912-S)

(*ii*) (*a*) $\cos x = \frac{4}{5}$
(*b*) $\cos x = -\frac{3}{5}$
(*c*) $\cos x = -\frac{1}{2}$ } between $-\pi$ and $x = \pi$ (*P.U.*)

(*iii*) $\tan x + \frac{1}{2}$ between $x = 0$ and $x = 2\pi$ (*P.U. 1939-S*)

(**Hints** : For solution of an equation graphically.

(*i*) Put each side of the equation equal to y and draw the graphs of the two equations obtained on the same axes.

(*ii*) Find what the abscissa of the point of intersection represents on the scale, taking 1 small division along the x-axis $= 10^\circ$. **Illustration** : In (*iii*) above put $y = \tan x$ and also $y = \frac{1}{2}$; draw the graph of $y = \tan x$ and then draw the graph of $y = \frac{1}{2}$, which is a straight line $\|$ to the x-axis. Mark their point of intersection ; drop $\perp$ from that point to the x-axis and measure the distance of its foot from the origin, taking 1 small division to be equal to 10°.]

6. Draw the graph of $\tan x$ as x varies from 0 to 2π and locate on the graph the values of x, for which

(*i*) $3\tan^2 x = 1$. (*P.U. 1946*)

(*ii*) $\tan x = \cot x$; (*P.U. 1946*)

Hint : $\tan x = \dfrac{1}{\tan x}$ or $\tan^2 x = 1$].

7. Draw the graph of $\tan\theta$ as θ varies from $-\pi$ to π and locate on the graph the values of θ for which

(i) $\tan\theta=3\cdot7$ (P.U. 1941-S)
(ii) $\tan x=-3$ (P.U.)

8. Draw the graph of $\sin x$ as x varies from $-\pi$ to $+\pi$ and locate on the graph the values of x for $\sin^2 x=\frac{1}{2}$.
(P.U. 1945)

9. Draw the graph of $y=\cos x$, where x lies between $-90°$ to $+180°$. Also solve graphically $\cos x=\frac{1}{3}$. (P.U. 1949)

10. Draw the graph of $\sin x$ as x varies from $-\pi$ to $+2\pi$. Solve graphically $\text{cosec}^2 x=1$. (P.U. 1947)

11. Trace the variation in $\cos x$ as x varies from 0 to 2π and draw the graph. Solve graphically $\cos^2 x=\frac{1}{4}$. (P.U. 1947-S)

MISCELLANEOUS EXERCISES ON CHAPTER V

1. Draw the graph of $\cos\theta$ from $\theta=-\pi$ to $\theta=\pi$. From the graph show that

(i) $\cos(-\theta)=\cos\theta$.
(ii) $\cos(\pi-\theta)=-\cos\theta$. (P.U.)

2. From the graph of $\sin x$, deduce that if $\sin x=k$, then there are two values of x between $0°$ to $360°$ for which $\sin x=k$, provided 'k' is a proper fraction.

Verify from the graph that $\sin(180°+45°)=-\sin 45°$.
(P.U.)

3. From the graphs of $\tan x$, and $\sin x$ show that for

$$0<x<\frac{\pi}{2},\ \sin x<x<\tan x. \qquad \text{(P.U. 1933)}$$

[**Hint:** x is measured in radians. Draw the graph of $y=\tan x$; $y=x$ radians; $y=\sin x$ and show that none of the graphs overlap, and that the graph of $y=x$ lies between the graphs of $x=\tan x$ and $y=\sin x$ and that the graph of $y=\sin x$ is the lowest of all.]

4. With the same axes of reference draw the graph of $y=\sin x$ and $y=\cot x$ from $x=0$ to $x=\pi$; for what acute angle is $\sin x=\cot x$. (P.U. 1938-S)

[**Hint:** Find the point of intersection and read the angle by drawing a $\perp$ from that point on the x-axis.]

5. Draw the graphs of $\sin 2x$ and $\tan x$ with the same axes and from your graphs read off the roots of the equation $\sin 2x = \tan x$ for values of x between 0 to 2π. (*P.U. 1935*)

6. Solve graphically the equations :

(*i*) $\cos x = x$ between $x = 0^\circ$ and $x = 360^\circ$. (*P.U. 1938*)

(*ii*) $\tan x = x$ between $x = 0$ and $x = \frac{\pi}{2}$. (*C.U.*)

[**Hint** : Whenever x is involved in such equations x is always measured in radians. Thus to draw the graph of $y = x$, we find that (0, 0) point lies on the graph ;

Also putting $x = 30^\circ = \frac{\pi}{6}$, we find that $y = \frac{\pi}{6} = \frac{3 \cdot 14}{6} = \cdot 52$

$\therefore$ the other point on the graph is $\left(\frac{\pi}{6}, \cdot 52\right)$.]

7. Trace the changes in the value of $\cos 2\theta$ as θ varies from 0° to 360° and draw the graph. (*P.U.*

[**Hint** : To trace the changes, put $2\theta = 90^\circ$ $\therefore$ $\theta = 45^\circ$; Hence θ increases from 0° to 35° ; from 45° to 90° ; from 90° to 135° and from 135° to 180°. As θ changes from 180° to 360°, the above changes are repeated. To draw the graph, find the value of y corresponding to the values of x differing by 15°.]

[$\therefore$ $2\theta = 30^\circ$]

8. Determine graphically the roots of the equation, $\tan x = 2 - \frac{4}{\pi} x$ which lie between 0 and π, x being in radians.

(*P.U. 1934*)

[**Hint** : Draw the graphs of $y = \tan x$ and $y = 2 - \frac{4}{\pi} x$ and read the X-co-ordinate of the point of intersection. For the graph of $y = 2 - \frac{4}{\pi} x$, put $x = 0$, then $y = 2$; again put $x = \frac{\pi}{2}$, then $y = 0$ $\therefore$ the two points (0, 2) and $\left(\frac{\pi}{2}, 0\right)$ lie on the graph]

9. Solve graphically the equation $\tan x = \cos x$ between $x = 0$ and $x = \frac{\pi}{2}$.

ANSWERS TO EXERCISES IN CHAPTER V

1. As θ increases from -180° to -90°, $\sin\theta$ is $-$ve and changes from 0 to -1; as θ increases from -90° to 0, $\sin\theta$ is $-$ve and changes from -1 to 0; as θ increases from 0 to $+90^\circ$ $\sin\theta$ is $+$ve and changes from 0 to $+1$; as θ increases from $+90^\circ$ to $+180^\circ$ $\sin\theta$ is $+$ve and changes from $+1$ to 0.

2. As x increases from -180° to -90°, $\cos x$ is $-$ve and changes from -1 to 0; as x increases from 90° to 0, $\cos x$ is $+$ve and changes from 0 to 1; as x increases from 0 to 90°, $\cos x$ is $+$ve and changes from 1 to 0; as x increases from 90° to 180°, $\cos x$ is $-$ve and changes from 0 to -1.

3. As θ increases from -360° to -270°, $\sec\theta$ is positive and increases from 1 to ∞; as θ passes through -270°, $\sec\theta$ suddenly changes from $+\infty$ to $-\infty$; as θ increases from -270° to -180°, $\sec\theta$ is negative and increases algebraically from $-\infty$ to -1; as θ increases from -180° to -90°, $\sec\theta$ is negative and decreases algebraically from -1 to $-\infty$; as θ passes through -90°, $\sec\theta$ suddenly changes from $-\infty$ to $+\infty$; as θ increases from -90° to 0, $\sec\theta$ is positive and decreases from ∞ to 1; as θ increases from 0° to 360° by intervals of 90°, the above changes are repeated.

4. As θ increases from -360° to -270°, $\operatorname{cosec}\theta$ is positive and decreases from ∞ to 1; as θ increases from -270° to -180°, $\operatorname{cosec}\theta$ is positive and increases from 1 to ∞; as θ passes through 180°, $\operatorname{cosec}\theta$ suddenly changes from $+\infty$ to $-\infty$; as θ increases from -180° to -90°, $\operatorname{cosec}\theta$ is negative and changes from $-\infty$ to -1; as θ increases from -90° to 0°, $\operatorname{cosec}\theta$ is negative and changes from -1 to $-\infty$; as θ increases from 0° to 360° by intervals of 90°, the above changes are repeated, 2π.

5. (*i*) (*a*) 53°, 125°. (*b*) -37°, -143°.
(*ii*) (*a*) $\pm 37^\circ$. (*b*) $\pm 127^\circ$. (*c*) $\pm 60^\circ$. (*iii*) 27°, 207°.

6. (*i*) $x=30^\circ, 210^\circ, 150^\circ, 330^\circ$,
(*ii*) $x=45^\circ, 135^\circ, 225^\circ, 315^\circ$.

7. (*i*) -105°, 75°. (*ii*) -73°, $+108^\circ$.

8 45°, 135°, -45°, -135°. **9.** Nearly 70°.

10. -90°, 90°, 270°. **11.** 60°, 300°, 120° and 240°.

Miscellaneous Examples on Chapter V

4. 53°. 5. $0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$.

6. (*i*) 42° (or ·73 radians). (*ii*) 0, 16° (or ·28 radians).

7. As θ increases from 0° to 45°, $\cos 2\theta$ is +ve and decreases from 1 to 0; as θ increases from 45° to 90°, $\cos 2\theta$ is −ve and changes from 0 to −1; as θ increases from 90° to 135°, $\cos 2\theta$ is −ve and changes from −1 to 0; as θ increases from 135° to 180°, $\cos 2\theta$ is +ve and increases from 0 to 1; as θ increases from 180° to 360° by intervals of 45°, the above changes are repeated.

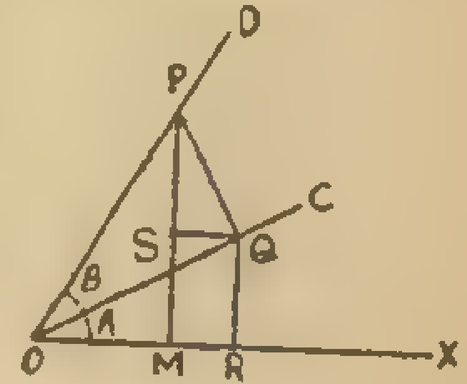
8. $\frac{\pi}{4}, \frac{3\pi}{4}$ 9. 38°.

CHAPTER VI

ADDITION AND SUBTRACTION THEOREMS

6·1. Addition Theorems.

To prove that



(*i*) $\sin(A+B)=\sin A\cos B+\cos A\sin B.$ (*P.U.* 1940, 42, 47)

(*ii*) $\cos(A+B)=\cos A\cos B-\sin A\sin B.$ (*P.U.* 1921, 27, 45, 47-S)

(*iii*) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$ (*P.U.* 1945, 1950)

Let the revolving line start from the initial position OX and trace out the angle XOC = A (say) and then trace out further the angle COD = B (say). Take any point P on the final position OD and draw PM and PQ perpendicular to the lines OX and OC respectively. From Q draw QS perpendicular to MP.

Then since the angles OQP and OMP are right angles, the quadrilateral OMQP is cyclic, and

$$\angle SPQ=\angle XOC=A$$

(*i*) $\sin(A+B)=\sin\angle XOP=\dfrac{MP}{OP}=\dfrac{MS+SP}{OP}=\dfrac{RQ+SP}{OP}$

$$=\frac{RQ}{OP}+\frac{SP}{OP}=\frac{RQ}{OQ}\cdot\frac{OQ}{OP}+\frac{SP}{PQ}\cdot\frac{PQ}{OP}\ *$$

$$=\sin A\cos B+\cos A\sin B.$$

(*ii*) $\cos(A+B)=\cos\angle XOP=\dfrac{OM}{OP}=\dfrac{OR-MR}{OP}=\dfrac{OR-SQ}{OP}$

* See hints on next page.

$$=\frac{OR}{OP}-\frac{SQ}{OP}=\frac{OR}{OQ}\cdot\frac{OQ}{OP}-\frac{SQ}{PQ}\cdot\frac{PQ}{OP}$$

$$=\cos A\cos B-\sin A\sin B.$$

(*iii*) $\tan(A+B)=\tan\angle XOP=\frac{MP}{OM}=\frac{MS-SP}{OR-MR}=\frac{RQ-SP}{OR-SQ}$

$$=\frac{\dagger\frac{RQ}{OR}+\frac{SP}{OR}}{1-\frac{SQ}{OR}}=\frac{\tan A+\frac{SP}{OR}}{1-\frac{SQ}{SP}\cdot\frac{SP}{OR}}$$

$$=\frac{\tan A+\frac{SP}{OR}}{1-\tan A.\frac{SP}{OR}}$$

Now since triangle PSQ and ORQ are similar

$$\frac{SP}{PQ}=\frac{OR}{OQ}\text{ or }\frac{SP}{OR}=\frac{PQ}{OQ}=\tan B$$

$$\therefore\ \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

---

*Note this step : we have $\frac{RQ}{OP}=\frac{RQ}{\text{something}}\cdot\frac{\text{something}}{OP}$. This 'something' is the hypotenuse of the $\triangle ORQ$, because sine of an angle has hypotenuse as the denominator. Also $\frac{SP}{OP}=\frac{SP}{PQ}\cdot\frac{PQ}{OP}$ $\because$ in this case PQ is the hypotenuse and cosine of an angle has also hypotenuse as the denominator.

Similarly $\frac{OR}{OP}=\frac{OR}{OQ}\cdot\frac{OQ}{OP}$ $\therefore$ OQ is the hypotenuse of the $\triangle$ of which QR is the base.

$\therefore$ The general rule for sine and cosine is that we should divide and multiply by the hypotenuse of the $\triangle$ of which the "length" in the numerator is base or $\perp$.

†Note we divide RQ by OR, $\because$ OR is the base and RQ is the $\perp$ of the $\angle$ ORQ. We are 'here' aiming at getting tan A or tan B. Also $\frac{SQ}{OR}=\frac{SQ}{SP}\cdot\frac{SP}{OR}$; we divide SQ by SP for SP is the base of the $\triangle$SPQ.

Hence rule for tangent is that we should divide and multiply by the base of the triangle.

Cor. 1. Alternative method for proving the formula

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

$$\tan(A+B)=\frac{\sin(A+B)}{\cos(A+B)}=\frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}.$$

Divide the numerator and denominator by cos A cos B

$$\therefore \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}.$$

Cor. 2. To prove that $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}$

$$\cot(A+B)=\frac{\cos(A+B)}{\sin(A+B)}=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}$$

$$=\frac{\cot A\cot B-1}{\cot B+\cot A}.$$

[by dividing the numerator and denominator by sin A sin B.

Cor. 3. By putting B=A

$\sin 2A=2\sin A\cos A.$

$\cos 2A=\cos^2 A-\sin^2 A=1-2\sin^2 A=2\cos^2 A-1$

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}.$$

Example 1. *To find the sine, cosine and tan of 75°.* (*D.U.*)

$\sin 75°=\sin(45°+30°)=\sin 45°\cos 30°+\cos 45°\sin 30°$

$$=\frac{1}{\sqrt2}\cdot\frac{\sqrt3}{2}+\frac{1}{\sqrt2}\cdot\frac12=\frac{\sqrt3+1}{2\sqrt2}$$

$\cos 75°=\cos(45°+30°)=\cos 45°\cos 30°-\sin 45°\sin 30°$

$$=\frac{1}{\sqrt2}\cdot\frac{\sqrt3}{2}-\frac{1}{\sqrt2}\cdot\frac12=\frac{\sqrt3-1}{2\sqrt2}.$$

$\tan 75°=\tan(45°+30°)=\dfrac{\tan 45°+\tan 30°}{1-\tan 45°\tan 30°}$ (*P.U. 1950*)

$$=\frac{1+\frac{1}{\sqrt3}}{1-\frac{1}{\sqrt3}}=\frac{\sqrt3+1}{\sqrt3-1}=\frac{\sqrt3+1}{\sqrt3-1}\times\frac{\sqrt3+1}{\sqrt3-1}=\frac{4+2\sqrt3}{3-1}$$

$=2+\sqrt3=3{\cdot}732.$

Example 2. *To find sin 105° and cos 105°.* (*D.U. 1936*)

$\sin 105°=\sin(60°+45°)=\sin 60°\cos 45°+\cos 60°\sin 45°$

$$=\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}+\tfrac{1}{2}\cdot\frac{1}{\sqrt{2}}=\frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$\cos 105^\circ=\cos(60^\circ+45^\circ)=\cos 60^\circ\cos 45^\circ-\sin 60^\circ\sin 45^\circ$$
$$=\tfrac{1}{2}\cdot\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}=-\frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Example 3. *Prove that* $\tan\left(\frac{\pi}{4}+A\right)=\frac{1+\tan A}{1-\tan A}$.

$$\tan\left(\frac{\pi}{4}+A\right)=\frac{\tan\frac{\pi}{4}+\tan A}{1-\tan\frac{\pi}{4}\tan A}=\frac{1+\tan A}{1-\tan A}.$$

Example 4. (*a*) *If* $A+B=\frac{\pi}{4}$ *prove that*

$$(1+\tan A)(1+\tan B)=2.$$

(*b*) *Deduce that* $22\tfrac{1}{2}^\circ=\sqrt{2}-1$. (*P.U. 1936*)

(*a*) $\tan(A+B)=\tan\frac{\pi}{4}=1.$

$\therefore\quad \frac{\tan A+\tan B}{1-\tan A\tan B}=1.$

or $\tan A+\tan B=1-\tan A\tan B.$

$\therefore\quad 1+\tan A+\tan B+\tan A\tan B=2.$

or $(1+\tan A)(1+\tan B)=2.$

(*b*) Put $A=B=22\tfrac{1}{2}^\circ \qquad \therefore\ A+B=45^\circ$

$\therefore\quad (1+\tan 22\tfrac{1}{2}^\circ)^2=2.$

or $\tan 22\tfrac{1}{2}^\circ=\sqrt{2}-1.$

Example 5. $\sin(90^\circ+\theta)=\sin 90^\circ.\cos\theta+\cos 90^\circ\sin\theta$
$=1\cos\theta+0.\sin\theta=\cos\theta.$

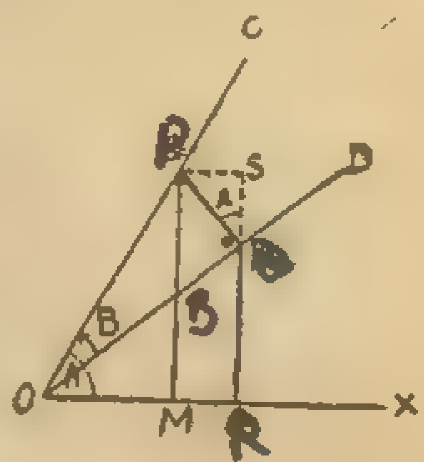
6·2. Subtraction Theorems.

To prove that

(*i*) $\sin(A-B)$
$=\sin A\cos B-\cos A\sin B$ (*P.U. 1946, 49*)

(*ii*) $\cos(A-B)$
$=\cos A\cos B+\sin A\sin B.$ (*P.U. 1941*)

(*iii*) $\tan(A-B)$
$=\frac{\tan A-\tan B}{1+\tan A\tan B}.$ (*C.U. 1912*)



Let the revolving line start from the initial position OX and trace out the $\angle XOB = A$ (say) and then revolve in the opposite direction from OC through the angle $COD = B$ (say). Then the angle XOD is the angle $A - B$. Take any point P on the final position OD and draw PM and PQ perpendicular to the lines OX and OC respectively. From Q draw QS perpendicular to MP produced, and $QR \perp OX$.

Then $\angle SPQ = \angle XOC = A$.

[$\because$ OMPQ is a cyclic quadrilateral]

(*i*) $$\sin(A - B) = \frac{MP}{OP} = \frac{MS - PS}{OP} = \frac{RQ - PS}{OP} = \frac{RQ}{OP} - \frac{PS}{OP}$$

$$= \frac{RQ}{OQ} \cdot \frac{OQ}{OP} - \frac{PS}{PQ} \cdot \frac{PQ}{OP}$$

$$= \sin A \cos B - \cos A \sin B.$$

(*ii*) $$\cos(A - B) = \frac{OM}{OP} = \frac{OR + RM}{OP} = \frac{OR + QS}{OP} = \frac{OR}{OP} + \frac{QS}{OP}$$

$$= \frac{OR}{OQ} \cdot \frac{OQ}{OP} + \frac{QS}{QP} \cdot \frac{QP}{OP}$$

$$= \cos A \cos B + \sin A \sin B.$$

(*iii*) $$\tan(A - B) = \frac{MP}{OM} = \frac{MS - PS}{OR + RM} = \frac{RQ - PS}{OR + QS} = \frac{\dfrac{RQ}{OR} - \dfrac{PS}{OR}}{1 + \dfrac{QS}{OR}}$$

$$= \frac{\tan A - \dfrac{PS}{OR}}{1 + \dfrac{QS}{PS} \cdot \dfrac{PS}{OR}} = \frac{\tan A - \dfrac{PS}{OR}}{1 + \tan A \cdot \dfrac{PS}{OR}}$$

Now since Δs PSQ and ORQ are similar

$$\frac{PS}{PQ} = \frac{OR}{OQ} \text{ or } \frac{PS}{OR} = \frac{PQ}{OR} = \tan B$$

$$\therefore \quad \tan(A - B) = \frac{\tan A - \tan + B}{1 + \tan A \tan B}$$

Cor. 1. Alternative proof for $\tan(A - B)$

$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$$

$$=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

[By dividing the numerator and denominator by cos A cos B.]

Cor. 2. To prove that $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$

$$\cot(A-B)=\frac{\cos(A-B)}{\sin(A-B)}=\frac{\cos A\cos B+\sin A\sin B}{\sin A\cos B-\cos A\sin B}$$

$$=\frac{\cot A\cot B+1}{\cot B-\cot A}$$

[By dividing the numerator and denominator by sin A sin B.]

Example 1. *To find sine, cosine and tan of 15°.*

(P.U. 1945)

$$\sin 15^\circ=\sin(45^\circ-30^\circ)=\sin 45^\circ\cos 30^\circ-\cos 45^\circ\sin 30^\circ$$

$$=\frac{1}{\sqrt2}\cdot\frac{\sqrt3}{2}-\frac{1}{\sqrt2}\cdot\frac{1}{2}=\frac{\sqrt3-1}{2\sqrt2}$$

$$\cos 15^\circ=\cos(45^\circ-30^\circ)=\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ$$

$$=\frac{1}{\sqrt2}\cdot\frac{\sqrt3}{2}+\frac{1}{\sqrt2}\cdot\frac{1}{2}=\frac{\sqrt3+1}{2\sqrt2}$$

$$\tan 15^\circ=\tan(45^\circ-30^\circ)=\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}$$

$$=\frac{1-\frac{1}{\sqrt3}}{1+\frac{1}{\sqrt3}}=\frac{\sqrt3-1}{\sqrt3+1}=\frac{\sqrt3-1}{\sqrt3+1}\times\frac{\sqrt3-1}{\sqrt3-1}=\frac{4-2\sqrt3}{3-1}$$

$$=2-\sqrt3.$$

Example 2. *To prove that* $\tan\left(\frac{\pi}{4}-A\right)=\dfrac{1-\tan A}{1+\tan A}$

$$\tan\left(\frac{\pi}{4}-A\right)=\frac{\tan\frac{\pi}{4}-\tan A}{1+\tan\frac{\pi}{4}\cdot\tan A}=\frac{1-\tan A}{1+\tan A}$$

Example 3 *Simplify* $\sin A+\cot B\cos A$

$$\sin A+\cot B\cos A=\sin A+\frac{\cos B}{\sin B}\cdot\cos A$$

$$=\frac{\sin A\sin B+\cos A\cos B}{\sin B}=\frac{\cos(A-B)}{\sin B}.$$

Example 4. *Prove that* $\cot A\ \frac{\sqrt{3}}{2}\sin B=2\cos\left(A+\frac{\pi}{3}\right)$

$$\cos A-\sqrt{3}\sin B=2\left(\frac{1}{2}\sin A-\frac{\sqrt{3}}{2}\sin A\right)$$

$$=2\left(\cos A\cos\frac{\pi}{3}-\sin A\sin\frac{\pi}{3}\right)$$

$$=2\cos\left(A+\frac{\pi}{3}\right).$$

Note 1. We have proved Arts. 1 and 3 on the assumption that A, B and the angles $A\pm B$ are acute and positive; but it can be shown that the results hold for all values of A and B, positive or negative.

Let $A'=A+\frac{\pi}{2}$

Then $\sin(A'+B)=\sin\left(A+B+\frac{\pi}{2}\right)=\cos(A+B)$

$$=\cos A\cos B-\sin A\sin B$$

$$=\sin A+\frac{\pi}{2}\Big)\cos B+\cos\left(A+\frac{\pi}{2}\right)\sin B.$$

$$=\sin A'\cos B+\cos A'\sin B.$$

In this case A' is obtuse and B is acute and the formula is still true.

Similarly we can take A acute and B obtuse by increasing B by $\frac{\pi}{2}$.

The formula can be similarly extended by increasing A' by $\frac{\pi}{2}$ and therefore A by π, and further by $\frac{3\pi}{2}$ and so on.

Note 2. The previous results can be summarised as follows :—

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\cot(A \pm B) = \frac{\cot A \cot B \mp 1}{\cot B \pm \cot A}.$$

6·3. To prove that

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B, = \cos^2 B - \cos^2 A$ *(P.U. 1941)*

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B, = \cos^2 B - \sin^2 A$ *(P.U. 1941)*

(i) $\sin(A+B)\sin(A-B) = (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$

$= (\sin^2 A \cos^2 B - \cos^2 A \sin^2 B)$

$= \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 B$

or $= (1 - \cos^2 A) - (1 - \cos^2 B) = \cos^2 B - \cos^2 A.$

(ii) $\cos(A+B)\cos(A-B)$

$= (\cos A \cos B - \sin A \sin B)(\cos A \cos B + \sin A \sin B)$

$= \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$

$= \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B$

$= \cos^2 A - \sin^2 B$

$= (1 - \sin^2 A) - (1 - \cos^2 B) = \cos^2 B - \sin^2 A.$

Example 1. *Prove that* $\sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right) = \frac{1}{\sqrt{2}} \sin A$

Since $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

$$\therefore \quad \sin^2\left(\frac{\pi}{8} + \frac{A}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{A}{2}\right)$$

$$= \sin\left(\frac{\pi}{8} + \frac{A}{2} + \frac{\pi}{8} - \frac{A}{2}\right)\sin\left(\frac{\pi}{8} + \frac{A}{2} - \frac{\pi}{8} + \frac{A}{2}\right)$$

$$= \sin\frac{\pi}{4} \cdot \sin A = \frac{1}{\sqrt{2}} \sin A$$

EXERCISE VI (A)

1. Find the value of $\sin 22^\circ \cos 38^\circ + \cos 22^\circ \sin 38^\circ$ *(D.U. 1943)*

[**Hint** : Apply $\sin(A+B) = \sin A \cos B + \cos A \sin B$.]

2. Prove that

(*i*) $\sin 40^\circ . \cos 10^\circ - \cos 40^\circ . \sin 10^\circ = \frac{1}{2}$

(*ii*) $\cos 35^\circ . \cos 25^\circ - \sin 35^\circ \sin 25^\circ = \frac{1}{2}$

3. Prove that

$$\cos(45^\circ - A).\cos(45^\circ - B) - \sin(45^\circ - A).\sin(45^\circ - B) = \sin(A+B)$$

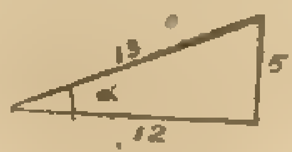
4. If $\sin \alpha = \frac{5}{13}$ and $\cos \beta = \frac{15}{17}$, find the values of (*i*) $\sin(\alpha - \beta)$ and (*ii*) $\cos(\alpha + \beta)$

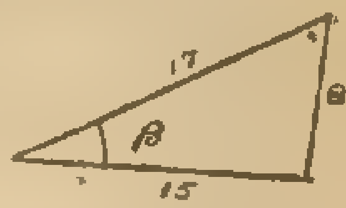
[**Hint** : (*ii*) $\sin \alpha = \frac{5}{13}$

$\therefore \cos \alpha = \frac{12}{13}$

Also $\cos \beta = \frac{15}{17}$

$\therefore \sin \beta = \frac{8}{17}$





$\therefore \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta = \frac{12}{13} . \frac{15}{17} - \frac{5}{13} . \frac{8}{17}$
$= \frac{140}{221}$; we ignore here the $-$ve sign of $\cos \alpha$ and $\cos \beta$, in which case $\cos(\alpha + \beta)$ will have four values The values are $+$ve if α and β are both acute angles.]

5. If $\cos A = \frac{1}{7}$ and $\cos B = \frac{13}{14}$ (A, B being acute), prove that $A - B = 60^\circ$. *(P.U.)*

[**Hint** : Show that $\cos A - B = \frac{1}{2}$ or $\sin(A - B) = \frac{\sqrt{3}}{2}$]

6. If $\sin A = \frac{1}{\sqrt{5}}$ and $\sin B = \frac{1}{\sqrt{10}}$ prove that $A + B = 45^\circ$

7. If $\tan A = \frac{4}{3}$ and $\tan B = 1$, find the avlue of $\tan(A - B)$.

8. If $\tan \alpha = \frac{5}{6}$ and $\tan \beta = \frac{1}{11}$, show that $\alpha = 45^\circ - \beta$.

9. If ABC is a rt. ∠d △ at C, prove that

$$\tan\left(A - \frac{\pi}{4}\right) = \frac{a-b}{a+b}.$$

[**Hint** : Expand $\tan\left(A - \frac{\pi}{4}\right)$ and put down the value of

$$\tan A = \frac{a}{b}\Big]$$

10. Prove that

$$\frac{\sin(A-B)}{\sin A \sin B}+\frac{\sin(B-C)}{\sin B \sin C}+\frac{\sin(C-A)}{\sin C \sin A}=0$$

[Hint : 1st term of the expression is $=\dfrac{\sin A \cos B-\cos A \sin B}{\sin A \sin B}$

$=\cot B-\cot A$ etc.]

11. Prove that

(i) $$\frac{\sin 3A \cos A+\cos 3A \sin A}{\cos^2 2A-\sin^2 2A}=\tan 4A$$

(ii) $$\tan^2 A-\tan^2 B=\frac{\sin(A+B).\sin(A-B)}{\cos^2 A \cos^2 B}$$

12. Show that (i) $$\frac{\tan(A-B)-\tan B}{1-\tan(A-B).\tan B}=\tan A$$ (P. U.)

(ii) $$\frac{\tan 69^\circ+\tan 66^\circ}{1-\tan 69^\circ.\tan 66^\circ}=-1.$$ (P U. 1936-S)

13. Show that $\cos\beta.\cos(2\alpha-\beta)=\cos^2\alpha-\sin^2(\alpha-\beta)$

(P U. 1944)

[Hint : Use $\cos^2 A-\sin^2 B=\cos(A+B).\cos(A-B)$]

14. Show that $\sec\left(\dfrac{\pi}{4}+\theta\right).\sec\left(\dfrac{\pi}{4}-\theta\right)=2\sec 2\theta.$

(P.U. 1941)

[Hint : L.H.S. $$=\frac{1}{\cos\left(\dfrac{\pi}{4}+\theta\right).\cos\left(\dfrac{\pi}{4}-\theta\right)}$$

$$=\frac{1}{\cos^2\dfrac{\pi}{4}-\sin^2\theta}=\frac{1}{\frac{1}{2}-\sin^2\theta}$$

$$=\frac{2}{1-2\sin^2\theta}=\frac{2}{\cos 2\theta}=2\sec 2\theta.]$$

Trigonometric ratios of $A+B+C$.

6·4. To prove that

(i) $\sin(A+B+C)=\cos A \cos B.\cos C$
$(\tan A+\tan B+\tan C-\tan A \tan B \tan C)$

(ii) $\cos(A+B+C)=\cos A \cos B.\cos C$
$(1-\tan B \tan C-\tan C.\tan A-\tan A \tan B)$

(iii) $\tan(A+B+C)$

$$=\frac{\tan A+\tan B+\tan C-\tan A \tan B \tan C}{1-\tan B \tan C-\tan C.\tan A-\tan A.\tan B}$$

Proof. (*i*) We have $\sin(A+\overline{B+C})$

$$=\sin A.\cos(B+C)+\cos A\sin(\overline{B+C})$$
$$=\sin A[\cos B.\cos C-\sin B\sin C]$$
$$+\cos A[\sin B.\cos C+\cos B\sin C]$$
$$=\sin A.\cos B.\cos C-\sin A.\sin B\sin C$$
$$+\cos A\sin B.\cos C+\cos A.\cos B.\sin C$$
$$=\cos A.\cos B.\cos C\left[\frac{\sin A}{\cos A}-\frac{\sin A\sin B\sin C}{\cos A.\cos B.\cos C}+\frac{\sin B}{\cos B}+\frac{\sin C}{\cos C}\right]$$
$$=\cos A.\cos B.\cos C[\tan A+\tan B+\tan C-\tan A.\tan B.\tan C].$$

(*ii*) $\cos(A+\overline{B+C})$

$$=\cos A.\cos(B+C)-\sin A\sin(\overline{B+C})$$
$$=\cos A(\cos B.\cos C-\sin B\sin C)$$
$$-\sin A(\sin B.\cos C+\cos B\sin C)$$
$$=\cos A\cos B.\cos C-\cos A\sin B\sin C$$
$$-\sin A\sin B\cos C-\sin A.\cos B.\sin C$$
$$=\cos A.\cos B.\cos C\left(1-\frac{\sin B.\sin C}{\cos B.\cos C}-\frac{\cos A.\cos B}{\sin A.\sin B}-\frac{\text{sis } A.\sin C}{\cos A.\cos C}\right)$$
$$=\cos A.\cos B\cos C(1-\tan B.\tan C-\tan A\tan B-\tan C.\tan A).$$

(*iii*) $\tan(A+B+C)=\dfrac{\sin(A+B+C)}{\cos(A+B+C)}.$

$$=\frac{\cos A\cos B\cos C(\tan A+\tan B+\tan C-\tan A\tan B\tan C)}{\cos A\cos B\cos C(1-\tan A\tan B-\tan B\tan C-\tan C\tan A)}$$
$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}.$$

Otherwise $\tan(A+\overline{B}+C)=\dfrac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$

$$=\frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\dfrac{\tan A+\tan B}{1-\tan A\tan B}.\tan C}$$
$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}.$$

Cor. 1. If A, B, C are the angles of a triangle, prove that $\tan A+\tan B+\tan C=\tan A \tan B \tan C$.

(*P.U.* 1946)

Now $\dfrac{\tan A+\tan B+\tan C-\tan A \tan B \tan C}{1-\tan A \tan B-\tan B \tan C-\tan C \tan A}$

$$=\tan(A+B+C)=\tan 180°=0$$

$\therefore$ $\tan A+\tan B+\tan C=\tan A \tan B \tan C$.

Cor. 2. If $A+B+C=90°$ prove that
$\tan A \tan B+\tan B \tan C+\tan C \tan A=1$.

Now $\dfrac{\tan A+\tan B+\tan C-\tan A \tan B \tan C}{1-\tan A \tan B-\tan B \tan C-\tan C \tan A}$

$$=\tan(A+B+C)=\tan 90°=\infty$$

$\therefore$ Denominator of L.H.S.$=0$.

$\therefore$ $1=\tan A \tan B+\tan B \tan C+\tan C \tan A$.

Cor. 3. From Cor. 1 if $A=B=C=60°$
$3 \tan 60°=\tan^3 60°$ or $\tan^2 60°=3$.

Hence $\tan 60°=\sqrt{3}$. (*P.U.* 1939)

Cor. 4. If $A=B=C$ in the above results, deduce that

$$\sin 3A=3\sin A-4\sin^3 A$$
$$\cos 3A=4\cos^3 A-3\cos A$$
$$\tan 3A=\frac{3\tan A-\tan^3 A}{1-3\tan^2 A}$$

(*P.U.* 1946)

Example 1. *Prove that $\tan 9\theta-\tan 6\theta-\tan 3\theta$*
$=\tan 9\theta \tan 6\theta \tan 3\theta$.

$$\tan 9\theta=\tan(6\theta+3\theta)=\frac{\tan 6\theta+\tan 3\theta}{1-\tan 6\theta \tan 3\theta}$$

$\therefore$ $\tan 9\theta-\tan 9\theta \tan 6\theta \tan 3\theta=\tan 6\theta+\tan 3\theta$

$\therefore$ $\tan 9\theta-\tan 6\theta-\tan 3\theta=\tan 9\theta \tan 6\theta \tan 3\theta$.

Example 2. *If A, B, C be the angles of a triangle, prove that*

$$\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$$

$\because$ $A+B+C=180°$ $\therefore$ $A+B=180-C$

$$\tan\left(\frac{A+B}{2}\right)=\tan\left(90°-\frac{C}{2}\right)=\cot\frac{C}{2}$$

$$\therefore \quad \frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\frac{1}{\tan\frac{C}{2}}$$

$$\therefore \tan\frac{A}{2}\tan\frac{C}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{A}{2}\tan\frac{B}{2}=1.$$

Example 3. *Show that $\tan 70°=2\tan 50°+\tan 20°$.*

$$\tan 70°=\tan(50+20)$$

$$=\frac{\tan 50°+\tan 20°}{1-\tan 50°\cdot\tan 20°}.$$

Cross-multiplying

$$\tan 70°-\tan 70°.\tan 50°.\tan 20°=\tan 50°+\tan 20°$$

Now $\tan 70°=\tan(90°-20°)=\cot 20°$

$$\therefore \tan 70°-\cot 20°.\tan 50°.\tan 20°=\tan 50°+\tan 20°$$

or $\quad \tan 70°-\tan 50°=\tan 50°+\tan 20°$

$[\because \quad \tan 20°.\cot 20=1]$

or $\qquad \tan 70°=2\tan 50°+\tan 20°.$

Example 4. *An angle α is divided into two parts, so that the ratio of the tangents of the parts is k, If x be the difference between two prats, show that $\sin x=\frac{k-1}{k+1}\sin\alpha$*

(P.U. 1930-S).

Let the two parts be A, B, so that $A+B=\alpha$ and $A-B=x$.

Now $\frac{\tan A}{\tan B}=k$ (give)

$\therefore$ By componendo and dividendo we have

$$\frac{\tan A-\tan B}{\tan A+\tan B}=\frac{k-1}{k+1} \quad \text{or} \quad \frac{\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}}{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}=\frac{k-1}{k+1}$$

or $$\frac{\sin A\cos B-\cos A\sin B}{\sin A\cos B+\cos A\sin B}=\frac{k-1}{k+1}$$

or $$\frac{\sin(A-B)}{\sin(A+B)}=\frac{k-1}{k+1} \quad \text{or} \quad \frac{\sin x}{\sin\alpha}=\frac{k-1}{k+1}$$

$$\therefore \sin x=\frac{k-1}{k+1}\sin\alpha.$$

EXERCISE VI (B)

1. Show that

$$\tan(B-C)+\tan(C-A)+\tan(A-B)$$
$$=\tan(B-C)\tan(C-A)\tan(A-B). \quad (\textit{P.U. 1935})$$

[Hint : $(B-C)+(C-A)=(B-A)=-(A-B)$

Take tangents of both sides

$$\therefore \quad \tan[(B-C)+(C-A)]=\tan[-(A-B)]$$
$$=-\tan(A-B)$$

Expand the L. H. S. and cross-multiply.]

2. Show that $\tan 3A.\tan 2A.\tan A=\tan 3A-\tan 2A-\tan A$ (*Pat. 1937*)

[Hint : $\tan 3A=\tan(2A+A)$ and expand.]

3. Show that $\tan 13A-\tan 9A-\tan 4A$
$=\tan 13A.\tan 9A.\tan 4A.$ (*P.U.*)

4. Prove that $\tan 15^\circ+\tan 30^\circ+\tan 15^\circ.\tan 30^\circ=1.$ (*P.U.*)

[Hint : $\tan(15^\circ+30^\circ)=\tan 45^\circ=1.$]

5. Prove that $\tan 50^\circ=\tan 40^\circ+2\tan 10^\circ.$

6. Show that $\dfrac{\cos 11^\circ+\sin 11^\circ}{\cos 11^\circ-\sin 11^\circ}=\tan 56^\circ.$ (*P.U. 1930-S*)

[Hint : Divide the numerator and denominator by $\cos 11^\circ$

$$\therefore \quad \text{Expr.}=\frac{1+\tan 11^\circ}{1-\tan 11^\circ}=\frac{\tan 45^\circ+\tan 11^\circ}{1-\tan 45^\circ.\tan 11^\circ}.]$$

7. If $\dfrac{\sin(\theta+\alpha)}{\sin(\theta-\alpha)}=\lambda$, prove that $\tan\theta=\dfrac{\lambda+1}{\lambda-1}\tan\alpha.$

[Hint : Apply dividendo and componendo.]

8. If $\tan(45^\circ-A)=n\cot(45^\circ-B)$, prove that

$$\frac{\sin(A+B)}{\cos(A-B)}=\frac{1-n}{1+n}.$$

9. If $\tan A=y\tan B$, prove that

$$\sin(A-B)=\frac{y-1}{y+1}.\sin\alpha, \text{ where } \alpha=A+B.$$

10. If $\tan\beta=\dfrac{n\sin\alpha.\cos\alpha}{1-n\sin^2\alpha}$ show that

$$\tan(\alpha-\beta)=(1-n)\tan\alpha.$$

[Sol. As the answer is a relation between $\tan(\alpha-\beta)$ and $\tan\alpha$ $\therefore$ we express $\tan\beta$ in terms of $\tan\alpha$.

Now $\tan\beta=\dfrac{\dfrac{n\sin\alpha\cos\alpha}{\cos^2\alpha}}{\dfrac{1-n\sin^2\alpha}{\cos^2\alpha}}$

[by multiplying and dividing by $\cos^2\alpha$]

$$=\frac{n\tan\alpha}{\sec^2\alpha-n\tan^2\alpha}$$

$$=\frac{n\tan\alpha}{1+(1-n)\tan^2\alpha}.$$

Now $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$$=\frac{\tan\alpha-\dfrac{n\tan\alpha}{1+(1-n)\tan^2\alpha}}{1+\dfrac{n\tan^2\alpha}{1+(1-n)\tan^2\alpha}}$$

$$=\frac{\tan\alpha+(1-n)\tan^3\alpha-n\tan\alpha}{1+\tan^2\alpha}$$

$$=\frac{(1-n)\tan\alpha\,(1+\tan^2\alpha)}{1+\tan^2\alpha}=\overline{1-n}\tan\alpha.$$

11. Show that

$$\frac{\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)}{\tan\left(\frac{\pi}{4}+\theta\right)-\tan\left(\frac{\pi}{4}-\theta\right)}=\operatorname{cosec}2\theta. \qquad (P.U.\ 1938)$$

12. Prove that $\cot A-\operatorname{cosec}2A=\cot 2A$. (P.U.)

13. Prove that $\dfrac{1-\tan(n+1)A\times\tan(n-1)A}{1+\tan(n+1)A\times\tan(n-1)A}=\dfrac{\cos 2nA}{\cos 2A}$.

14. Show that $\tan\left(\dfrac{\pi}{4}+\dfrac{A}{2}\right)=\sec A+\tan A$. (P.U. (1939)

[L.H.S. $=\dfrac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}}=\dfrac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}\times\dfrac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}+\sin\frac{A}{2}}$

$$=\frac{\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)^2}{\cos^2\frac{A}{2}-\sin^2\frac{A}{2}}=\frac{1+2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos A}$$

$$=\frac{1+\sin A}{\cos A}=\sec A+\tan A.]$$

MISCELLANEOUS EXERCISES ON CHAPTER VI

1. Prove that (*i*) $\sin\theta+\cos\theta=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$.

(*ii*) $\cos 5\theta.\cos 2\theta+\sin 5\theta.\sin 2\theta=\cos 2\theta.\cos\theta-\sin 2\theta\sin\theta$.

(*iii*) $[\sin\theta\,(1+\sin\theta)+\cos\theta\,(1+\cos\theta)]\times[\sin\theta\,(1-\sin\theta)$
$+\cos\theta\,(1-\cos\theta)]=\sin 2\theta$. (*P.U. 1939*)

2. Prove that

(*i*) $\sin^2(A+B)-\sin^2(A-B)=\sin 2A\sin 2B$.

(*ii*) $\cos^2\left(\frac{\pi}{4}-\theta\right)-\sin^2\left(\frac{\pi}{4}-\phi\right)=\sin(\theta+\phi)\cos(\theta-\phi)$.

(*iii*) $\sin^2\left(\frac{\pi}{4}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{4}-\frac{A}{2}\right)=\sin A$.

3. Prove that

$$\sec(A+B)=\frac{\sec A\sec B\operatorname{cosec} A\operatorname{cosec} B}{\operatorname{cosec} A.\operatorname{cosec} B-\sec A.\sec B}$$

(*P.U. 1936*

[**Hint** : Express the R.H.S. in terms of sines and cosines and the result follows]

4. In a quadrilateral ABCD, show that
$\cos A.\cos B-\cos C.\cos D=\sin A\sin B-\sin C.\sin D$.

[**Hint** : $A+B+C+D=360°$. It should be noted that products of cos A and cos B and that of sin A and sin B occur together and similarly *t*-ratios for C and D occur together we, ∴ have $A+B=360°-(C+D)$ and take cosine of both sides.]

5. If $A+B=45°$, prove that

(*i*) $(1+\tan A)(1+\tan B)=2$

(*ii*) $(\cot A-1)(\cot B-1)=2$. (*P.U. 1936*)

[**Hint** : (*ii*) In (*i*) put $\tan A=\dfrac{1}{\cot A}$ and $\tan B=\dfrac{1}{\cot B}$

we get $(\cot A+1)(\cot B+1)=2\cot A\cot B$

$\therefore\quad \cot A.\cot B+\cot A+\cot B+1=2\cot A\cot B$

or $\quad \cot A.\cot B-\cot A-\cot B+1=2$

or $\quad (\cot A-1)(\cot B-1)=2.$]

6. Express $\cos(A-B+C)$ in terms of sines and cosines. (*P.U.*)

7. Show that (*i*) $\tan\left(45^\circ+\dfrac{\alpha}{2}\right)+\tan\left(45^\circ-\dfrac{\alpha}{2}\right)=2\sec\alpha$ (*P.U.*)

(*ii*) $\dfrac{1-\tan^2\left(\dfrac{\pi}{2}-x\right)}{1+\tan^2\left(\dfrac{\pi}{2}-x\right)}=\sin 2x$ (*P.U.*)

8. Show that

(*i*) $\dfrac{\tan^2 x+\tan^2 y}{1-\tan^2 x\tan^2 y}=\tan(x+y).\tan(x-y)$

(*ii*) $\dfrac{2}{\tan(45^\circ+A)+\tan(45^\circ-A)}=\cos 2A$ (*P.U.*

9. Show that

(*i*) $\dfrac{\sin 3A.\cos A+\cos 3A.\sin A}{\cos^2 2A-\sin^2 2A}=\tan 4A$ (*C.U.*)

(*ii*) $\cot\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right)=\sec x+\tan x$ (*D U. 1942*)

10. If $A+B+C=180^\circ$, prove that

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

(*P.U. 1942*)

[**Hint** : $\dfrac{B}{2}+\dfrac{C}{2}=\dfrac{180}{2}-\dfrac{A}{2}=90-\dfrac{A}{2}$

$\therefore\ \cot\left(\dfrac{B}{2}+\dfrac{C}{2}\right)=\cot\left(90-\dfrac{A}{2}\right)$

$$\text{or } \frac{\cot\frac{B}{2}\cdot\cot\frac{C}{2}-1}{\cot\frac{B}{2}+\cot\frac{C}{2}}=\tan\frac{A}{2}=\frac{1}{\cot\frac{A}{2}}$$

Cross-multiply.]

11. If $A+B+C=180°$, prove that

$\sin A.\sin(B-C)+\sin B.\sin(C-A)+\sin C.\sin(A-B)=0$

[**Hint :** $A=180-(B+C)$

$\therefore \quad \sin A=\sin(180-\overline{B+C})=\sin \overline{B+C}$

$\therefore$ 1st term of the expression is $\sin(B+C).\sin(B-C)$

$=\sin^2 B-\sin^2 C,\ldots\ldots$etc]

12. Prove that $\cos 2\theta \cos 2\phi+\sin^2(\theta-\phi)-\sin^2(\theta+\phi)$

$=\cos(2\theta+2\phi)$ *(P.U. 1944)*

[**Hint :** Use $\sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)$]

ANSWERS TO EXERCISES IN CHAPTER VI

Exercises VI A

1. $\frac{\sqrt{3}}{2}$. 4. $-\frac{21}{221}, \frac{140}{221}$. 7. $\frac{1}{7}$.

Miscellaneous Exercises on Chapter VI

13. $\cos A.\cos B.\cos C+\cos A.\sin B\sin C$

$-\cos B.\sin C.\sin A+\cos C\sin A\sin B.$

CHAPTER VII

TRANSFORMATION OF SUMS AND PRODUCTS

7·1. Transformation of products into sum or difference.

To prove that

(1) $2 \sin A \cos B = \sin (A+B) + \sin (A-B)$.
(2) $2 \cos A \sin B = \sin (A+B) - \sin (A-B)$.
(3) $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$.
(4) $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$.

We have already proved in the previous chapter that

$$\sin (A+B) = \sin A \cos B + \cos A \sin B \quad \dots \quad (i)$$
$$\sin (A-B) = \sin A \cos B - \cos A \sin B \quad \dots \quad (ii)$$
$$\cos (A+B) = \cos A \cos B - \sin A \sin B \quad \dots \quad (iii)$$
$$\cos (A-B) = \cos A \cos B + \sin A \sin B \quad \dots \quad (iv)$$

From (*i*) and (*ii*) by addition and subtraction.

$$\sin (A+B) + \sin (A-B) = 2 \sin A \cos B \quad \dots \quad (v)$$
$$\sin (A+B) - \sin (A-B) = 2 \cos A \sin B \quad \dots \quad (vi)$$

From (*iii*) and (*iv*) by addition and subtraction.

$$\cos (A+B) + \cos (A-B) = 2 \cos A \cos B \quad \dots \quad (vii)$$
$$\cos (A+B) - \cos (A-B) = -2 \sin A \sin B \quad \dots \quad (viii)$$

Writing the results (*v*) to (*viii*) in the reverse order we have

(1) $2 \sin A \cos B = \sin (A+B) + \sin (A-B)$.
(2) $2 \cos A \sin B = \sin (A+B) - \sin (A-B)$.
(3) $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$.
(4) $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$.

Note 1. The student should make himself perfectly familiar with these formulae ; they are often used in the next chapters and no progress is possible without their application.

Note 2. To remember the above results in words we have

(1) twice sine × cosine = sin (sum) + sin (diff.)
(2) twice cosine × sin = sin (sum) − sin (diff.)
(3) twice cosine × cosine = cos (diff.) + cos (sum).
(4) twice sine × sine = cos (diff.) − cos (sum).

Difference means first angle − second angle.

SOLVED EXAMPLES

Express as sum or difference.

1. $2\sin 5A\cdot\cos 3A = \sin(\text{sum}) + \sin(\text{diff})$
$= \sin(5A+2A) + \sin(5A-3A)$
$= \sin 8A + \sin 2A$

2. $2\cos 75^\circ.\cos 15^\circ = \cos(75^\circ - 15^\circ) + \cos(75^\circ + 15^\circ)$
$= \cos 60^\circ + \cos 90^\circ$
$= \frac{1}{2} + 0 = \frac{1}{2}.$

3. $2\sin 3\theta \sin 7\theta = \cos(3\theta - 7\theta) - \cos(3\theta + 7\theta)$
$= \cos(-4\theta) - \cos 10\theta$
$= \cos 4\theta - \cos 10\theta$

4. $\cos\frac{\theta}{2}\sin\frac{3\theta}{2} = \frac{1}{2}\left[2\cos\frac{\theta}{2}\cdot\sin\frac{3\theta}{2}\right]$
$= \frac{1}{2}\left[\sin\left(\frac{\theta}{2} + \frac{3\theta}{2}\right) - \sin\left(\frac{\theta}{2} - \frac{3\theta}{2}\right)\right]$
$= \frac{1}{2}[\sin 2\theta - \sin(-\theta)]$
$= \frac{1}{2}[\sin 2\theta + \sin\theta].$

5. $\sin\left(\frac{\pi}{4} - \theta\right).\sin\left(\frac{\pi}{4} + \theta\right)$

$= \frac{1}{2}\left[2\sin\left(\frac{\pi}{4} - \theta\right).\sin\left(\frac{\pi}{4} + \theta\right)\right]$

$= \frac{1}{2}\left[\cos\left(\frac{\pi}{4} - \theta - \frac{\pi}{4} - \theta\right) - \cos\left(\frac{\pi}{4} - \theta + \frac{\pi}{4} + \theta\right)\right]$

$= \frac{1}{2}\left[\cos(-2\theta) - \cos\frac{\pi}{4}\right] = \frac{1}{2}\cos 2\theta$

6. $\cos(A+B).\cos(A-B)$
$= \frac{1}{2}[2\cos(A+B).\cos(A-B)]$
$= \frac{1}{2}[\cos 2A + \cos 2B]$

7. Prove that $\cos 20^\circ \cos 40^\circ.\cos 60^\circ.\cos 80^\circ = \frac{1}{16}.$ (*P.U. 1949*)

L. H. S. $= \frac{1}{2}\cos 20^\circ.\cos 40^\circ.\cos 80^\circ$
$= \frac{1}{4}[(2\cos 20^\circ.\cos 40^\circ)\cos 80^\circ]$
$= \frac{1}{4}[\{\cos 60^\circ + \cos(20^\circ - 40^\circ)\}\cos 80^\circ]$
$= \frac{1}{4}[\{(\frac{1}{2} + \cos 20^\circ)\}\cos 80^\circ]$
$= \frac{1}{8}[\{(1 + 2\cos 20^\circ)\}\cos 80^\circ]$

$$=\tfrac{1}{8}\left[\cos 80^\circ+\overline{2\cos 20^\circ\cos 80^\circ}\right]$$
$$=\tfrac{1}{8}\left[\cos 80^\circ+\cos(20+80)+\cos(20-80^\circ)\right]$$
$$=\tfrac{1}{8}\left[\cos 80^\circ+\cos 100^\circ+\cos 60^\circ\right]$$
$$=\tfrac{1}{8}\left[\cos 80^\circ+\cos(180^\circ-80^\circ)+\tfrac{1}{2}\right]$$
$$=\tfrac{1}{8}\left[\cos 80^\circ-\cos 80^\circ+\tfrac{1}{2}\right]=\tfrac{1}{16}.$$

EXERCISES VII (A)

1. Express as sum or difference

(*i*) $2\sin 4\theta\cos 2\theta$, (*ii*) $\sin(\alpha+\beta).\sin(\alpha-\beta)$.

(*iii*) $2\cos(\alpha-\beta).\sin(\alpha+\beta)$.

2. Find the value of

(*i*) $\cos 75^\circ.\cos 15^\circ$, (*ii*) $\sin 22\tfrac{1}{2}^\circ\sin 67\tfrac{1}{2}^\circ$.

3. Prove that

$$2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$$

[**Hint:** $2\cos\frac{\pi}{13}\cdot\cos\frac{9\pi}{13}=\cos\frac{10\pi}{13}+\cos\frac{8\pi}{13}$

Now $\cos\frac{10\pi}{13}=\cos\left(\pi-\frac{3\pi}{13}\right)=-\cos\frac{3\pi}{13}$

$\cos\frac{8\pi}{13}=\cos\left(\pi-\frac{5\pi}{13}\right)=-\cos\frac{5\pi}{13}$]

4. Provide that (*i*) $\sin 20^\circ.\sin 40^\circ.\sin 60^\circ.\sin 80^\circ=\frac{3}{16}$.

(*P. U. 1940*)

(*ii*) $\sin 20^\circ.\sin 40^\circ.\sin 80^\circ.\sin 90^\circ=\frac{\sqrt{3}}{8}$. (*P. U. 1947*)

5. If A, B, C are the angles of a triangle, prove that

$\sin A.\sin\overline{B-C}+\sin B\sin(C-A)+\sin C\sin(A-B)=0$.

[**Hint:** 1st term $=\tfrac{1}{2}\,2\sin A.\sin\overline{B-C}$

$=\tfrac{1}{2}\left[2\sin(B+C).\sin\overline{B-C}\right]$

$(\because\ \sin A=\sin(180-\overline{B+C})=\sin\overline{B+C}$

$=\tfrac{1}{2}\left[\cos(2C)-\cos 2B\right]$.

6. Prove that

$\cos(A+B)+\sin(A-B)=2\sin(45^\circ+A).\cos(45^\circ+B)$

(*P. U. 1943*)

[**Hint:** R. H. S. $=\sin(45^\circ+A+45^\circ+B)$

$+\sin(45^\circ+A-45^\circ-B)=\sin(90+\overline{A+B})+\sin(A-B)$]

7. Prove that $4 \sin \alpha . \sin\left(\alpha+\frac{\pi}{3}\right) . \sin\left(\alpha+\frac{2\pi}{3}\right)=\sin 3\alpha$.

(P. U.)

[**Hint :** $\text{L. H. S.}=2 \sin \alpha . 2 \sin\left(\alpha+\frac{\pi}{3}\right) . \sin\left(\alpha+\frac{2\pi}{3}\right)$

$$=2 \sin \alpha . \left[\cos \frac{\pi}{3}-\cos(\pi+2\alpha)\right]$$
$$=2 \sin \alpha \left[\tfrac{1}{2}+\cos 2\alpha\right]$$
$$=\sin \alpha+2 \sin \alpha \cos 2\alpha$$
$$=\sin \alpha+\sin(\alpha-2\alpha)+\sin(\alpha+2\alpha)$$
$$=\sin \alpha+\sin 3\alpha-\sin \alpha].$$

8. Prove that

$\cos^2 A+\cos^2 B-2 \cos A . \cos B . \cos(A+B)=\sin^2(A+B)$

(P. U. 1943-S)

[**Hint :** 3rd term $=-2 \cos A \cos B . \cos(A+B)$

$$=-[\cos(A+B)+\cos(A-B)] \cos(A+B)$$
$$=-\cos^2(A+B)-\cos^2 A+\sin^2 B].$$

7·2. To prove that

(1) $\sin C+\sin D=2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$

(P. U. 1943)

(2) $\sin C-\sin D=2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$

(3) $\cos C+\cos D=2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$

(4) $\cos C-\cos D=2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}$.

Since
$$\sin(A+B)=\sin A \cos B+\cos A \sin B \quad \ldots \quad (i)$$
$$\sin(A-B)=\sin A \cos B-\cos A \sin B \quad \ldots \quad (ii)$$
$$\cos(A+B)=\cos A \cos B-\sin A \sin B \quad \ldots \quad (iii)$$
$$\cos(A-B)=\cos A \cos B+\sin A \sin B \quad \ldots \quad (iv)$$

From (*i*) and (*ii*) by addition and subtraction.
$$\sin(A+B)+\sin(A-B)=2 \sin A \cos B \quad \ldots \quad (v)$$
$$\sin(A+B)-\sin(A-B)=2 \cos A \sin B \quad \ldots \quad (vi)$$

From (*iii*) and (*iv*) by addition and subtraction
$$\cos(A+B)+\cos(A-B)=2 \cos A \cos B \quad \ldots \quad (vii)$$
$$\cos(A+B)-\cos(A-B)=2 \sin A \sin B \quad \ldots \quad (viii)$$

Now let $A+B=C$, and $A-B=D$;

Then $A=\frac{C+D}{2}$ and $B=\frac{C-D}{2}$.

Substituting in results (*v*) to (*viii*) we get

(1) $\sin C+\sin D=2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(2) $\sin C-\sin D=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(3) $\cos C+\cos D=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(4) $\cos C-\cos D=2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$.

Note. To remember the results in words we have

Sum of two sines=twice sin (half sum) cos (half diff.)

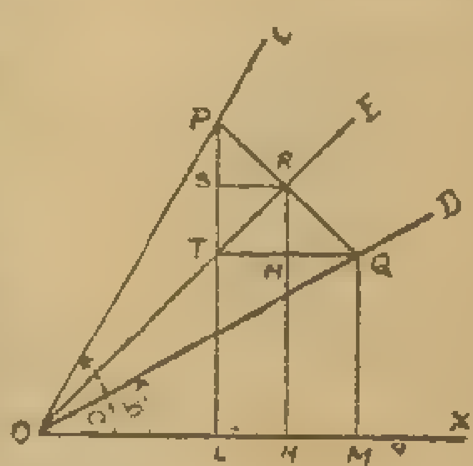
Diff. of two sines=twice cos (half sum) sin (half diff.)

Sum of two consines=twice cos (half sum) cos (half diff.)

Diff. of two consines=twice sine (half sum) sin (half diff. reversed).

Please note the last result carefully, where half-diff. reversed means *second angle-first angle*.

7·3. Geometrical Proofs.*



The results of the last article can be proved geometrically.

Let $\angle XOC=C$ and $XOD=D$. Bisect $\angle DOC$ by OE.

Let R be any point on OE and draw PRQ perpendicular to OR meeting OC in P and OD on Q. Draw PL, QM and RN perpendicular to OX and RS and QT perpendicular to PL QT cutting RN in H.

Then $\angle QOR=\angle ROP=\frac{1}{2}\angle QOP=\frac{C-D}{2}$

$$\angle XOR=\angle XOQ+\angle QOR=D+\frac{C-D}{2}=\frac{C+D}{2}.$$

Since R is the middle point of PQ and we have

(*i*) $OP=OQ=1$ unit of length (say).
(*ii*) $PL+QM=2\,RN$.
(*iii*) $PT=2\,RH$; $TQ=2TH$.
(*iv*) $OL+OM=2\,ON$.

Now (1) $\sin C+\sin D=LP+MQ=2NR$

$$=2\,OR\sin\angle XOR=2\cos\angle ROP.\sin\angle XOR$$

$$=2\sin\frac{C+D}{2}.\cos\frac{C-D}{2}.$$

(2) $\sin C-\sin D=LP-MQ=TP=2HR=2\,\frac{HR}{RQ}.\frac{RQ}{1}$

$$=2\cos\frac{C+D}{2}\sin\frac{C-D}{2}.$$

$\because$ $\angle XOR$ and $\angle HPQ$ are complement of $\angle ORN$.

(3) $\cos C+\cos D=OL+OM=2\,ON=2\,OR\cos\angle NOR$

$$=2\,OP.\cos\angle ROP\cos\angle NOR$$

$$=2\cos\frac{C+D}{2}\cos\frac{C-D}{2}.$$

(4) $\cos D-\cos C=OM-OL=LM=2NM=\frac{NM}{RQ}.RQ$

$$=2\sin\frac{C+D}{2}.\sin\frac{C-D}{2}.$$

SOLVED EXERCISES

1. $\sin 5\theta+\sin 3\theta=2\sin\frac{5\theta+3\theta}{2}\cos\frac{5\theta+3\theta}{2}=2\sin 4\theta\cos\theta.$

2. $\sin 15\theta-\sin 7\theta=2\cos\frac{15\theta+7\theta}{2}\sin\frac{15\theta-7\theta}{2}$

$$=2\cos 11\theta\sin 4\theta.$$

3. $\cos 3\theta+\cos 7\theta=2\cos\frac{3\theta+7\theta}{2}.\cos\frac{7\theta-3\theta}{2}$

$$=2\cos 5\theta\cos 2\theta.$$

4. $\cos 3\theta-\cos 7\theta=2\sin\frac{3\theta+7\theta}{2}\sin\frac{7\theta-3\theta}{2}$

$$=2\sin 5\theta\sin 2\theta.$$

5. $\cos 20^\circ-\cos 80^\circ=2\sin\frac{20^\circ+80^\circ}{2}\sin\frac{80^\circ-20^\circ}{2}$

$$=2\sin 50^\circ\sin 30^\circ=\sin 50^\circ.$$

6. Express $\sin 7A+\sin 3A$ and $\cos 3A-\cos 7A$ as products.

Sol. (i) $\sin 7A+\sin 3A=2\sin\frac{7A+3A}{2}\cos\frac{7A-3A}{2}$

$=2\sin 5A\cos 2A.$

(ii) $\cos 3A-\cos 7A=2\sin\frac{3A+7A}{2}\sin\frac{7A-3A}{2}$

$=2\sin 5A\sin 2A.$

7. Express $1+\sin A$ as the product of a sine and cosine.

Sol. $1+\sin A=\sin 90°+\sin A$

$=2\sin\frac{90°+A}{2}\cos\frac{90°-A}{2}$

$=2\sin\left(45°+\frac{A}{2}\right)\cos\left(45°-\frac{A}{2}\right).$

8. Prove that $\sin 50°+\sin 10°-\sin 70°=0.$ (*D.U. 1938*

Sol. $\sin 50°+\sin 10°-\sin 70°$

$=2\sin\frac{50°+10°}{2}\cos\frac{50°-10°}{2}-\sin(90°-20°)$

$=2\sin 30°\cos 20°-\cos 20°=\cos 20°-\cos 20°=0.$

9 Prove that $\frac{\sin A+\sin B}{\cos A+\cos B}=\tan\frac{A+B}{2}.$ (*P.U*)

L.H.S. $=\frac{\sin A+\sin B}{\cos A+\cos B}=\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}$

$=\tan\frac{A+B}{2}.$

10 Prove that $\frac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}=\tan 4A.$

(*P.U.*)

L.H.S. $=\frac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}$

$=\frac{(\sin A+\sin 7A)+(\sin 3A+\sin 5A)}{(\cos A+\cos 7A)+(\cos 3A+\cos 5A)}$

$=\frac{2\sin 4A\cos 3A+2\sin 4A\cos A}{2\cos 4A\cos 3A+2\cos 4A\cos A}$

$=\frac{2\sin 4A(\cos 3A+\cos A)}{2\cos 4A(\cos 3A+\cos A)}=\tan 4A.$

11. If $\sin\theta = n\sin(\theta+2\alpha)$, prove that

$$\tan(\theta+\alpha)=\frac{1+n}{1-n}\tan\alpha. \qquad \textit{(P.U. 1938)}$$

Sol. $\dfrac{\sin(\theta+2\alpha)}{\sin\theta}=\dfrac{1}{n}$.

By dividendo and componendo

$$\therefore \quad \frac{\sin(\theta+2\alpha)+\sin\theta}{\sin(\theta+2\alpha)-\sin\theta}=\frac{1+n}{1-n}$$

or $$\frac{2\sin(\theta+\alpha)\cos\alpha}{2\cos(\theta+\alpha)\sin\alpha}=\frac{1+n}{1-n}$$

$$\therefore \quad \tan(\theta+\alpha)=\frac{1+n}{1-n}\tan\alpha.$$

EXAMPLES VII (B)

1. Express in the form of a product

(*i*) $\sin 24°-\sin 76°$. (*P.U.*)

(*ii*) $\cos 2\theta-\cos 4\theta$. (*C.U.*)

2. Prove that (*i*) $\sin 51°+\cos 81°=\cos 21°$. (*P.U.*)

(*ii*) $\cos 20°\ \cos 100°+\cos 140°=0$. (*D.U. 1937*)

Hint : (*i*) If there is a sum or difference of sine and cosine, we can express them as a product if we use

$$\cos\theta=\sin(90°-\theta) \text{ or } \sin\theta=\cos(90°-\theta)$$

$$\begin{aligned}\therefore \quad \sin 51°+\cos 81° &= \sin 51°+\sin(90°-81°)\\ &= \sin 51°+\sin 9°\\ &= 2\sin\frac{51°+9°}{2}.\cos\frac{51°-9°}{2}\\ &= 2\sin 30°.\cos 21°=\cos 21°.\end{aligned}$$

(*ii*) $\cos 20°+\cos 100°+\cos 140°=2\cos 60°.\cos 40°+\cos 140°$
$=\cos 40°+\cos 140°=\cos(180°-140°)+\cos 140°$
$=-\cos 140°+\cos 140°=0.$

3. Prove that (*i*) $\cos 52°+\cos 68°+\cos 172°=0$.

(*ii*) $\sin 10°+\sin 20°+\sin 40°+\sin 50°=\sin 70°+\sin 80°$.

[Hint : (*ii*) L.H.S.

$=(\sin 10°+\sin 50°)+(\sin 20°+\sin 40°)$
$=2\sin 30°\cos 20°+2\sin 30°.\cos 10°$
$=2\sin 30°[\cos 20°+\cos 10°]=\cos 20°+\cos 10°$
$=\sin(90°-20°)+\sin(90°-10°)=\sin 70°+\sin 80°$]

4. Show that (*i*) $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\dfrac{\tan\left(\dfrac{A+B}{2}\right)}{\tan\left(\dfrac{A-B}{2}\right)}$. (*P.U. 1943*)

(*ii*) $\dfrac{\sin A+\sin 2A+\sin 4A+\sin 5A}{\cos A+\cos 2A+\cos 4A+\cos 5A}=\tan 3A$

5. Show that (*i*) $\cos A+\cos(120°-A)+\cos(120°+A)=0$

(*ii*) $\sin A+\sin\left(A+\dfrac{2\pi}{3}\right)+\sin\left(A+\dfrac{4\pi}{4}\right)=0.$ (*C.U.*)

6. Prove that (*i*) $\dfrac{\sin 8\theta\cos\theta-\cos 3\theta\sin 6\theta}{\cos 2\theta.\cos\theta-\sin 3\theta\sin 4\theta}=\tan 2\theta$

(*ii*) $\dfrac{\sin 11A\sin\ +\sin 7A\sin 3A}{\cos 11A.\sin\ +\cos 7A\sin 3A}=\tan 8A$ (*P.U.*)

7. Show that

(*i*) $\cos 3\theta\cos 2\theta+\sin 4\theta.\sin\theta=\cos 2\theta.$

(*ii*) $\sin\dfrac{11\theta}{4}.\sin\dfrac{\theta}{4}+\sin\dfrac{7\theta}{4}.\sin\dfrac{3\theta}{4}=\sin\theta\sin 2\theta.$ (*C.U.*)

8. If $\dfrac{\sin(A+B)}{\cos(A-B)}=\dfrac{1-\lambda}{1+\lambda}$, prove that

$$\tan\left(\frac{\pi}{4}-B\right)=\lambda.\cot\left(\frac{\pi}{4}-A\right).$$

9. Show that

(*i*) $\dfrac{\sin A+\sin(A+B)+\sin(A+2B)}{\cos A+\cos(A+B)+\cos(A+2B)}=\tan(A+B)$

(*ii*) $\dfrac{\sin(A-B)+\sin A+\sin(A+B)}{\cos(A-B)+\cos A+\cos(A+B)}=\tan A.$

MISCELLANEOUS EXERCISE ON CHAPTER VII

1. Prove that $(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=4\sin^2\dfrac{\alpha-\beta}{2}.$

2. Prove that

$4\cos\alpha\cos\beta\cdot\cos\gamma=\cos\overline{\alpha+\beta+\gamma}+\cos(\beta+\gamma-\alpha)$
$+\cos(\gamma+\alpha-\beta)+\cos(\alpha+\beta+\gamma)$ (*P.U. 1943-S*)

Hint : R.H.S. $=[\cos(\alpha+\beta+\gamma)+\cos(\beta+\gamma+\alpha)]+$
$[\cos(\overline{\gamma+\alpha}-\beta)+\cos(\alpha+\beta-\gamma)]$
$=2\cos\overline{\beta+\gamma}\cdot\cos\alpha+2\cos\alpha\cos\overline{\gamma-\beta}.$
$=2\cos\alpha(\cos\overline{\beta+\gamma}+\cos\overline{\gamma-\beta})$ etc.]

3. Show that

$$\sin^2 A+\sin^2(A-B)-2\sin A\cos B.\sin(A-B)=\sin^2 B$$

[Hint : L.H.S.

$$=\sin^2 A+\sin(A-B)[\sin(A-B)-2\sin A.\cos B]$$
$$=\sin^2 A+\sin(A-B)[-\sin(A+B)]$$
$$=\sin^2 A-[(\sin^2 A-\sin^2 B)]=\sin^2 B].$$

4. Prove that

$$\sin(A-D)\sin(B-C)+\sin(B-D)\sin(C-A)$$
$$+\sin(C-D).\sin(A-B)=0.$$ *(A.U. 1936)*

5. If $A+B+C=180°$, prove that

$$\cos\frac{A}{2}.\cos\frac{B-C}{2}+\cos\frac{B}{2}\cos\frac{C-A}{2}+\cos\frac{C}{2}.\cos\frac{A-B}{2}$$
$$=\sin A+\sin B+\sin C.$$ *(P.U. 1943-S)*

[Hint : $\cos\frac{A}{2}=\cos\left(90-\frac{B+C}{2}\right)=\sin\frac{B+C}{2}$

Similarly $\cos\frac{B}{2}=\sin\frac{A+C}{2}$; $\cos\frac{C}{2}=\sin\frac{A+C}{2}$].

6. $\cos 2A+\cos 4A+\cos 6A+\cos 8A$

$$=4\cos A\cos 2A.\cos 5A.$$ *(C.U.)*

7. Prove that

(i) $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$$=4\cos\frac{\alpha+\beta}{2}.\cos\frac{\beta+\gamma}{2}.\cos\frac{\gamma+\alpha}{2}.$$ *(P.U.)*

(ii) $\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$

$$=4\sin\frac{\alpha+\beta}{2}.\sin\frac{\beta+\gamma}{2}.\sin\frac{\gamma+\alpha}{2}.$$

8. Prove that

(i) $\tan(A+30°)+\cot(A-30°)=\dfrac{1}{\sin 2A-\sin 60°}$

(D.U. 1939)

(ii) $\dfrac{1}{\tan 3\theta-\tan\theta}-\dfrac{1}{\cot 3\theta-\cot\theta}=\cot 2\theta.$

(iii) $\dfrac{1}{\tan 3\theta+\tan\theta}-\dfrac{1}{\cot 4\theta+\cot\theta}=\cot 4\theta.$

[Hint : Express tans and cots on the L.H.S. into sines and cosines and simplify.]

9. Prove that

$$\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)=\frac{1}{8}$$

L. H. S.

$$=\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)$$

$$=\left[\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\overline{\pi-\frac{\pi}{8}}\right)\right]\left[\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\overline{\pi-\frac{3\pi}{8}}\right)\right]$$

$$=\left[\left(1+\cos\frac{\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)\right]\left[\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\right]$$

$$=\left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right)$$

$$=\sin^2\frac{\pi}{8}\cdot\sin^2\frac{3\pi}{8}=\frac{1}{4}\left(2\sin\frac{\pi}{8}\cdot\sin\frac{3\pi}{8}\right)^2$$

$$=\frac{1}{4}\left(\cos\frac{\pi}{4}-\cos\frac{\pi}{2}\right)^2=\frac{1}{8}$$

10. Prove that

$$\frac{\sin(n+1)\theta+2\sin n\theta+\sin(n-1)\theta}{\cos(n-1)\theta-\cos(n+1)\theta}=\cot\frac{\theta}{2}$$

11. Prove that

$$\frac{\cos 11^\circ+\sin 11^\circ}{\cos 11^\circ-\sin 11^\circ}=\tan 56^\circ \qquad \textit{(P. U. 1930-S)}$$

[Sol. Expr. $=\dfrac{1+\dfrac{\sin 11^\circ}{\cos 11^\circ}}{1-\dfrac{\sin 11^\circ}{\cos 11^\circ}}=\dfrac{1+\tan 11^\circ}{1-\tan 11^\circ}$

$=\tan(45^\circ+11^\circ)=\tan 56^\circ.$

ANSWERS TO EXERCISES IN CHAPTER VII

Exercises VII (A)

1. (i) $\sin 6\theta+\sin 2\theta$. (ii) $\frac{1}{2}[\cos(2\beta)-\cos(2\alpha)]$
(iii) $\sin(2\alpha)+\sin(2\beta)$.

2. (i) $\dfrac{+\sqrt{3}}{4}$ (ii) $\dfrac{1}{2\sqrt{2}}$

Exercise VII (B)

1. (i) $-2\cos 50^\circ\cdot\sin 26^\circ$. (ii) $2\sin 3\theta\cdot\sin\theta$

CHAPTER VIII

TRIGONOMETRIC RATIOS OF MULTIPLE AND SUB-MULTIPLE ANGLES

8·1. Trigonometric ratios of 2A in terms of those of A.

To prove that

(*i*) $\sin 2A = 2 \sin A \cos A$

(*ii*) $\cos 2A = \cos^2 A - \sin^2 A \quad \ldots \quad (1)$

$= 1 - 2\sin^2 A \quad \ldots \quad (2)$

$= 2\cos^2 A - 1 \quad \ldots \quad (3)$

(*iii*) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

(*i*) $\sin 2A = \sin (A + A)$

$= \sin A \cos A + \cos A . \sin A = 2 \sin A . \cos A.$

(*ii*) $\cos 2A = \cos (A + A)$

$= \cos A . \cos A - \sin A . \sin A = \cos^2 A - \sin^2 A \quad (1)$

$= (1 - \sin^2 A) - \sin^2 A \quad = 1 - 2\sin^2 A \quad (2)$

$= \cos^2 A - (1 - \cos^2 A) \quad = 2\cos^2 A - 1 \quad (3)$

(*iii*) $\tan 2A = \dfrac{\tan A + \tan A}{1 - \tan A \tan A} = \dfrac{2 \tan A}{1 - \tan^2 A}.$

Cor: From formulas. (*ii*) 2 and 3 we get by transposition.

(1) $1 - \cos 2A = 2\sin^2 A$

(2) $1 + \cos 2A = 2\cos^2 A$

(3) and by dividing (1) by (2) we get $\dfrac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A$

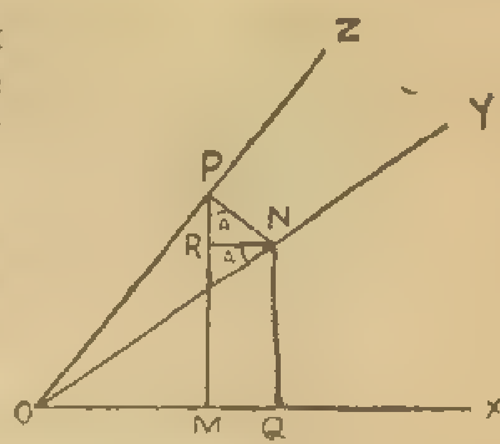
(*P. U.*)

8·2. To prove geometrically that

(*i*) $\sin 2A = 2 \sin A \cos A$

(*ii*) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$ (*P. U. 1943*)

(*iii*) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$



Let the revolving line trace out an angle $XOY = A$ and then revolve further through the same $\angle A$, so that angle $XOZ = 2A$.

Take a pt. P in the final position of the revolving line, and draw two $\perp$s PN and PM on OY and OX and NR on PM.

Then $\angle RPN = 90° - \angle PNR$

$= \angle RNO = \angle XOY = \angle A.$

$\therefore$ (*i*) $\sin 2A = \sin XOZ$

$$= \frac{MP}{OP} = \frac{MR + RP}{OP} = \frac{QN + RP}{OP} = \frac{QN}{OP} + \frac{RP}{OP}$$

$$= \frac{QN}{ON} \cdot \frac{ON}{OP} + \frac{RP}{NP} \cdot \frac{NP}{OP}$$

$= \sin A.\cos A + \cos A.\sin A = 2 \sin A.\cos A$

(*ii*) $\cos 2A = \cos XOZ = \frac{OM}{OP} = \frac{OQ - MQ}{OP} = \frac{OQ - RN}{OP}$

$$= \frac{OQ}{OP} - \frac{RN}{OP}$$

$$= \frac{OQ}{ON} \cdot \frac{ON}{OP} - \frac{RN}{NP} \cdot \frac{NP}{OP}$$

$= \cos A.\cos A - \sin A.\sin A$

$= \cos^2 A - \sin^2 A = 2 \cos^2 A - 1.$

(*iii*) $\tan 2A = \tan XOZ$

$$= \frac{MP}{OM} = \frac{MR + RP}{OQ - MQ} = \frac{QN + RP}{OQ - RN}$$

Dividing the numerator and denominator by OQ

we get $\tan 2A = \dfrac{\dfrac{QN}{OQ} + \dfrac{RP}{OQ}}{1 - \dfrac{RN}{OQ}} = \dfrac{\tan A + \dfrac{RP}{OQ}}{1 - \dfrac{RN}{RP} \cdot \dfrac{RQ}{OQ}} = \dfrac{\tan A + \dfrac{NP}{ON}}{1 - \tan A. \dfrac{NP}{ON}}$

[$\because$ Δs RPN and QON are similar $\therefore$ $\frac{RP}{OQ} = \frac{NP}{ON}$]

$$= \frac{\tan A + \tan A}{1 - \tan A.\tan A} = \frac{2 \tan A.}{1 - \tan^2 A}$$

[**Note.** The method can be easily remembered as it is the same as that for proving "A+B" formulae by making B=A.]

8'4. Examples.

1. If $\sin A = \frac{4}{5}$, find $\sin 2A$ and $\cos 4A$,

$\because \quad \cos A = \sqrt{1-\sin^2 A} = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$

$\therefore \quad \sin 2A = 2 \sin A \cos A = 2 \cdot \frac{4}{5} \cdot \frac{3}{5} = \frac{24}{25}$

$\cos 4A = 1 - 2 \sin^2 2A = 1 - 2\left(\frac{24}{25}\right)^2 = -\frac{527}{625}$

2. Find $\cos 22\frac{1}{2}°$ (*P. U. 1942*)

Now $2\cos^2 22\frac{1}{2} = 1 + \cos 45°$ $\quad [\because \ 2\cos^2 A = 1 + \cos 2A]$

$\therefore \quad \cos 22\frac{1}{2}° = \pm\sqrt{\frac{1+\cos 45}{2}} = +\sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}}$

$[\because \ \cos 22\frac{1}{2}° \text{ is } +\text{ve}]$

$$= +\sqrt{\left(\frac{1+\frac{\sqrt{2}}{2}}{2}\right)} = +\sqrt{\frac{2+\sqrt{2}}{4}} = +\frac{\sqrt{2+\sqrt{2}}}{2}$$

Example. *Find* $\sin 22\frac{1}{2}°$. (*P. U. 1942*)

3. Prove that $\tan\frac{A}{2} = \frac{\sin A}{1+\cos A} = \frac{1-\cos A}{\sin A}$

Now $\dfrac{\sin A}{1+\cos A} = \dfrac{2\sin\frac{A}{2}\cos\frac{A}{2}}{1+2\cos^2\frac{A}{2}-1} = \tan\dfrac{A}{2}$

Also $\dfrac{1-\cos A}{\sin A} = \dfrac{1-\left(1-2\sin^2\frac{A}{2}\right)}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \tan\dfrac{A}{2}$

Example. *Find* $\tan 22\frac{1}{2}°$ *and* $\cot 22\frac{1}{2}°$.

TWO IMPORTANT FORMULAE

8'5. To prove that $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$ (*P. U. 1949*)

$\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$ (*P. U. 1949*)

$$\sin 2A = 2 \sin A \cos A$$
$$1 = \cos^2 A + \sin^2 A$$

$$\therefore \sin 2A = \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A} = \frac{2 \dfrac{\sin A}{\cos A}}{1 + \dfrac{\sin^2 A}{\cos^2 A}} = \frac{2 \tan A}{1 + \tan^2 A}$$

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$1 = \cos^2 A + \sin^2 A$$

$$\cos 2A = \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \frac{1 - \dfrac{\sin^2 A}{\cos^2 A}}{1 + \dfrac{\sin^2 A}{\cos^2 A}} = \frac{1 - \tan^2 A}{1 + \tan^2 A}$$

Cor. Putting $2A = \theta$

$$\sin \theta = \frac{2 \tan \dfrac{\theta}{2}}{1 + \tan^2 \dfrac{\theta}{2}} \text{ and } \cos \theta = \frac{1 - \tan^2 \dfrac{\theta}{2}}{1 + \tan^2 \dfrac{\theta}{2}}$$

8·6. Trigonometric ratios of 3A in terms of thoseof A.

To prove that $\sin 3A = 3 \sin A - 4 \sin^3 A$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} \qquad (P.U.\ 1913)$$

$$\begin{aligned} \sin 3A &= \sin (A + 2A) = \sin A \cos 2A + \cos A \sin 2A \\ &= \sin A (1 - 2 \sin^2 A) + \cos A . 2 \sin A \cos A \\ &= \sin A - 2 \sin^3 A + 2 \sin A (1 - \sin^2 A) \\ &= 3 \sin A - 4 \sin^3 A \end{aligned}$$

$$\begin{aligned} \cos 3A &= \cos (A + 2A) = \cos A \cos 2A - \sin A \sin 2A \\ &= \cos A . (2 \cos^2 A - 1) - \sin A . 2 \sin A \cos A \\ &= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A) \\ &= 4 \cos^3 A - 3 \cos A \end{aligned}$$

$$\tan 3A = \tan (A + 2A) = \frac{\tan A + \tan 2A}{1 - \tan A . \tan 2A}$$

$$= \frac{\tan A + \dfrac{2 \tan A}{1 - \tan^2 A}}{1 - \tan A . \dfrac{2 \tan A}{1 - \tan^2 A}}$$

$$= \frac{\tan A (1 - \tan^2 A) + 2 \tan A}{(1 - \tan^2 A) - 2 \tan^2 A} = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

Example 1. *Deduce tan 3A from sin 3A and cos 3A.*

Example 2. *From the formula for tan 3A find the value of tan 30°.*

8·7. To prove that

$\sin 5A = 5\sin A - 20\sin^3 A + 16\sin^5 A$ (*P.U.*)

$\cos 5A = 5\cos A - 20\cos^3 A + 16\cos^5 A.$ (*P.U. 1944-S*)

$$\begin{aligned}
\sin 5A &= \sin(2A+3A)\\
&= \sin 2A\cos 3A + \cos 2A\sin 3A\\
&= 2\sin A\cos A\,(4\cos^3 A - 3\cos A) + (1-2\sin^2 A)(3\sin A - 4\sin^3 A)\\
&= 2\sin A\cos^2 A\,(4\cos^2 A - 3) + (1-2\sin^2 A)(3\sin A - 4\sin^3 A)\\
&= 2\sin A\,(1-\sin^2 A)(1-4\sin^2 A) + 3\sin A - 10\sin^3 A + 8\sin^5 A\\
&= 2\sin A - 10\sin^3 A + 8\sin^5 A + 3\sin A - 10\sin^3 A + 8\sin^5 A\\
&= 5\sin A - 20\sin^3 A + 16\sin^5 A
\end{aligned}$$

$$\begin{aligned}
\cos 5A &= \cos(2A+3A)\\
&= \cos 2A\cos 3A - \sin 2A\sin 3A\\
&= (2\cos^2 A - 1)(4\cos^3 A - 3\cos A) - (2\sin A\cos A)(3\sin A - 4\sin^3 A)\\
&= 8\cos^5 A - 10\cos^3 A + 3\cos A - 2\cos A\,(1-\cos^2 A)(4\cos^2 A - 1)\\
&= 8\cos^5 A - 10\cos^3 A + 3\cos A + 2\cos A - 10\cos^3 A + 8\cos^5 A\\
&= 5\cos A - 20\cos^3 A + 16\cos^5 A
\end{aligned}$$

Example 1. *Prove that*

$$\frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \tan A$$

$$\frac{\sin A + \sin 2A}{1 + \cos A + \cos 2A} = \frac{\sin A + 2\sin A\cos A}{1 + \cos A + 2\cos^2 A - 1}$$

$$= \frac{\sin A\,(1 + 2\cos A)}{\cos A\,(1 + 2\cos A)} = \frac{\sin A}{\cos A} = \tan A.$$

Example 2. *Express* $\dfrac{\sin 3A}{\sin 2A - \sin A}$ *in terms of cos A.*

$$\frac{\sin 3A}{\sin 2A - \sin A} = \frac{3\sin A - 4\sin^3 A}{2\sin A\cos A - \sin A}$$

$$=\frac{\sin A\{3-4(1-\cos^2 A)\}}{\sin A(2\cos A-1)}=\frac{4\cos^2 A-1}{2\cos A-1}$$

$$=\frac{(2\cos A+1)(2\cos A-1)}{2\cos A-1}$$

$$=2\cos A+1.$$

Example 3. *Prove that* $\cos 20^\circ \cos 40^\circ . \cos 60^\circ \cos 80^\circ=\frac{1}{16}$. *(P.U. 1940-49)*

1st Method. $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

$$=\frac{1}{2\sin 20^\circ}.2\sin 20^\circ \cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$$

$$=\frac{1}{2\sin 20^\circ}\sin 40^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$$

$$=\frac{1}{4\sin 20^\circ}\sin 80^\circ \cos 60^\circ \cos 80^\circ$$

$$=\frac{1}{8\sin 20^\circ}\sin 160^\circ \cos 60^\circ$$

$$=\frac{1}{8\sin 20^\circ}\sin(180^\circ-160^\circ).\cos 60^\circ$$

$$=\frac{1}{8\sin 20^\circ}\sin 20^\circ.\frac{1}{2}=\frac{1}{16}.$$

2nd Method. $\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ$

$$=\cos 20^\circ \cos(60^\circ-20^\circ).\cos(60^\circ+20^\circ).\cos 60^\circ$$
$$=\cos 20^\circ(\cos^2 60^\circ-\sin^2 20^\circ).\cos 60^\circ$$
$$=\cos 20^\circ.(\tfrac{1}{4}-1+\cos^2 20^\circ).\cos 60^\circ$$
$$=\cos 20^\circ(4\cos^2 20-3)\tfrac{1}{4}\cos 60^\circ$$
$$=(4\cos^3 20^\circ-3\cos 20^\circ)\tfrac{1}{4}\cos 60^\circ$$
$$=\cos 3.20^\circ.\tfrac{1}{4}\cos 60^\circ$$
$$=\tfrac{1}{4}\cos 60^\circ.\cos 60^\circ=\tfrac{1}{4}.\tfrac{1}{2}.\tfrac{1}{2}=\tfrac{1}{16}.$$

Example 4. *Prove that* $\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)=\sqrt{\frac{1+\sin A}{1-\sin A}}$
$=\sec A+\tan A.$ *(P.U.)*

$$\tan\left(\frac{\pi}{4}+\frac{A}{2}\right)=\frac{1+\tan\frac{A}{2}}{1-\tan\frac{A}{2}}$$

$$=\frac{1+\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}{1-\dfrac{\sin\frac{A}{2}}{\cos\frac{A}{2}}}=\sqrt{\left\{\frac{\cos\frac{A}{2}+\sin\frac{A}{2}}{\cos\frac{A}{2}-\sin\frac{A}{2}}\right\}^2}$$

$$=\sqrt{\frac{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2}+\sin^2\frac{A}{2}-2\sin\frac{A}{2}\cos\frac{A}{2}}}=\sqrt{\frac{1+\sin A}{1-\sin A}}$$

$$=\sqrt{\frac{(1+\sin A)(1+\sin A)}{(1-\sin A)(1+\sin A)}}=\frac{1+\sin A}{\cos A}=\sec A+\tan A.$$

Example 5. *Prove that $\cot A-\tan A=2\cot 2A$.*

(P.U. 1943)

$$\cot A-\tan A=\frac{1}{\tan A}-\tan A=\frac{1-\tan^2 A}{\tan A}$$

$$=\frac{2}{\dfrac{2\tan A}{1-\tan^2 A}}=\frac{2}{\tan 2A}=2\cot 2A.$$

Example 6. *Prove that* $\dfrac{\sin 3A}{\sin A}-\dfrac{\cos 3A}{\cos A}=2.$

1st Meteod. $\dfrac{\sin 3A}{\sin A}-\dfrac{\cos 3A}{\cos A}=\dfrac{\sin 3A\cos A-\cos 3A\sin A}{\sin A\cos A}$

$$=\frac{\sin(3A-A)}{\sin A\cos A}=\frac{\sin 2A}{\sin A\cos A}$$

$$=\frac{2\sin A\cos A}{\sin A\cos A}=2.$$

2nd Method. $\dfrac{\sin 3A}{\sin A}-\dfrac{\cos 3A}{\cos A}$

$$=\frac{3\sin A-4\sin^3 A}{\sin A}-\frac{4\cos^3 A-3\cos A}{\cos A}$$

$$=(3-4\sin^2 A)-(4\cos^2 A-3)=6-4(\sin^2 A+\cos^2 A)$$

$$=6-4=2.$$

Example 7. *Prove that*

$$\cos 2A=\frac{2}{\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)}. \qquad (D.U.)$$

$$\text{R. H. S.}=\frac{2}{\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)}$$

$$=\frac{2}{\frac{1+\tan A}{1-\tan A}+\frac{1-\tan A}{1+\tan A}}$$

$$=\frac{2(1-\tan^2 A)}{(1+\tan A)^2+(1-\tan A)^2}$$

$$=\frac{2(1-\tan^2 A)}{2(1+\tan^2 A)}=\cos 2A.$$

EXERCISES VIII (A)

1. If A is acute and $\sin A=\frac{12}{13}$, find the value of $\sin 2A$, $\cos 2A$.

2. If $\tan A=\frac{1}{3}$, find $\sin 2A$, $\cos 2A$, $\tan 2A$.

3. If $\tan \alpha=\frac{1}{5}$ and $\tan \beta=\frac{1}{239}$, show that $4\alpha=45^\circ+\beta$.

[**Hint :** Find the value of $\tan 4\alpha$ and $\tan (45^\circ+\beta)$ and prove that the two values are equal.]

4. If $\tan \theta=\dfrac{b}{a}$, find the value of $a \cos 2\theta+b \sin 2\theta$. *(D.U. 1939)*

5. If $\tan x=\frac{1}{3}$ find $\tan 3x$. *(P.U.)*

6. If $\sin x=\frac{3}{5}$, find the value of $\sin 3x$ and $\cos 3x$, given that x is obtuse :

Prove that following identities :

7. (*i*) $\dfrac{\sin 2A}{1+\cos 2A}=\tan A$ (*ii*) $\dfrac{\sin 2A}{1-\cos 2A}=\cot A.$

8. (*i*) $\dfrac{1-\cos A}{\sin A}=\tan\dfrac{A}{2}$ (*ii*) $\dfrac{1+\cos A}{\sin A}=\cot\dfrac{A}{2}$.

9. $\dfrac{\sin A+\sin 2A}{1+\cos A+\cos 2A}=\tan A.$

[Hint : $\text{L.H.S.} = \dfrac{\sin A + 2\sin A . \cos A}{\cos A + 2\cos^2 A}$

$$= \frac{\sin A\,(1+2\cos A)}{\cos A\,(1+2\cos A)} = \tan A.]$$

10. $\dfrac{1+\sin A}{1+\cos A} = \tfrac{1}{2}\left(1+\tan\dfrac{A}{2}\right)^2.$

[Hint : $\text{L.H.S.} = \dfrac{\cos^2\dfrac{A}{2} + \sin^2\dfrac{A}{2} + 2\sin\dfrac{A}{2}.\cos\dfrac{A}{2}}{2\cos^2\dfrac{A}{2}}$

$$= \frac{\left(\cos\dfrac{A}{2} + \sin\dfrac{A}{2}\right)^2}{2\cos^2\dfrac{A}{2}} \text{ etc.}]$$

11. (*i*) $\cot A - \tan A = 2\cot 2A.$ (*P.U. 1943-S*)

(*ii*) $\cot A + \tan A = 2\,\text{cosec}\,2A.$

[Hint : (*i*) $\text{L. H. S.} = \dfrac{\cos A}{\sin A} - \dfrac{\sin A}{\cos A} = \dfrac{\cos^2 A - \sin^2 A}{\sin A\cos A}$

$$= \frac{2\cos 2A}{\sin 2A}.]$$

12. (*i*) $\tan(45^\circ+\theta) + \tan(45^\circ-\theta) = 2\sec 2\theta.$ (*D U.*)

(*ii*) $\tan\left(\dfrac{\pi}{4}+\theta\right) - \tan\left(\dfrac{\pi}{4}-\theta\right) = 2\tan 2\theta.$

13. (*i*) $\sin 2\theta = \dfrac{\tan\left(\dfrac{\pi}{4}+\theta\right) - \tan\left(\dfrac{\pi}{4}-\theta\right)}{\tan\left(\dfrac{\pi}{4}+\theta\right) - \tan\left(\dfrac{\pi}{4}-\theta\right)}.$

(*ii*) $\sin 2\theta = \cos^2\left(\dfrac{\pi}{4}-\theta\right) - \sin^2\left(\dfrac{\pi}{4}-\theta\right).$

14. $\dfrac{1+\sin\theta}{1-\sin\theta} = \tan^2\left(\dfrac{\pi}{4}+\dfrac{\theta}{2}\right) = (\sec\theta + \tan\theta)^2.$

(*P.U. 1942-S*)

Hint : [L.H.S. $=\dfrac{\cos^2\dfrac{\theta}{2}+\sin^2\dfrac{\theta}{2}+2\sin\dfrac{\theta}{2}.\cos\dfrac{\theta}{2}}{\cos^2\dfrac{\theta}{2}+\sin^2\dfrac{\theta}{2}-2\sin\dfrac{\theta}{2}.\cos\dfrac{\theta}{2}}$

$$=\frac{\left(\cos\dfrac{\theta}{2}+\sin\dfrac{\theta}{2}\right)^2}{\left(\cos\dfrac{\theta}{2}-\sin\dfrac{\theta}{2}\right)^2}\text{etc.}$$

For second part put L. H. S. $=\dfrac{1+\sin\theta}{1-\sin\theta}\times\dfrac{1+\sin\theta}{1+\sin\theta}$

$$=\frac{(1+\sin\theta)^2}{1-\sin^2\theta}=\left(\frac{1+\sin\theta}{\cos\theta}\right)^2 \text{ etc.]}$$

15. (*i*) $4\sin A.\sin(60°+A)\sin(60°-A)=\sin 3A$. (*C.U.*)
(*ii*) $4\cos A.\cos(60°+A)\cos(60°-A)=\cos 3A$. (*P.U. 1948*)

16. (*i*) $\cos 20°.\cos 40°.\cos 60°.\cos 80°=\frac{1}{16}$. (*P.U. 1940, P.U. 1949*)

(*ii*) $\sin 20°.\sin 40°.\sin 60°.\sin 80°=\frac{3}{16}$. (*P.U. 1940-47*)

17. (*i*) $\tan A+\tan(60°+A)+\tan(120°+A)=3\tan 3A$. (*P.U. 1939-S*)

[**Hint :** Last two terms of the expression on L. H. S.
$$=\frac{\sqrt{3}+\tan A}{1-\sqrt{3}\tan A}+\frac{-\sqrt{3}+\tan A}{1+\sqrt{3}\tan A}=\frac{8\tan A}{1-3\tan^2 A}.]$$

(*ii*) $\cot A+\cot(60°+A)+\cot(60°-A)=3\cot 3A$.

18. $\operatorname{cosec} 2A=\cot A-\cot 2A=\tan A+\cot 2A$. (*P.U. 1937-S*)

19. $\sec A+\tan A=\dfrac{1+\tan\dfrac{A}{2}}{1-\tan\dfrac{A}{2}}$. (*P.U.*)

20. (*i*) $\operatorname{cosec} A+2\operatorname{cosec} 2A=\sec A\cot\dfrac{A}{2}$. (*P.U. 1941*)

(*ii*) $\tan\theta=\tan\dfrac{\theta}{2}+\frac{1}{2}\tan\theta.\sec^2\dfrac{\theta}{2}$.

21. $\sec\left(\dfrac{\pi}{4}+\theta\right).\sec\left(\dfrac{\pi}{4}-\theta\right)=2\sec 2\theta$. (*P.U. 1941*)

22. $[\sin\theta\,(1+\sin\theta)+\cos\theta\,(1+\cos\theta)]\times[\sin\theta\,(1-\sin\theta)$
$+\cos\theta\,(1-\cos\theta)]=\sin 2\theta.$ *(P.U. 1939)*

23. Show that $\sin 2A$ and $\tan A$ have always the same sign.

[**Hint :** $\sin 2A=\dfrac{2\tan A}{1+\tan^2 A}$. Now the denominator is the sum of two squares and is,..., always +ve : Hence the sign of $\sin 2A$ and $\tan A$ are always the same.]

24. Show that $(\cos\theta+\sin\theta)^4+(\cos\theta-\sin\theta)^4=3-\cos 4\theta.$
(P.U. 1942 Supp.)

2. TRIGONOMETRIC RATIOS OF $\frac{A}{2}$ IN TERMS OF A

8·7. To express

$\sin\dfrac{A}{2}$, $\cos\dfrac{A}{2}$ and $\tan\dfrac{A}{2}$ in terms of $\cos A$.

Since $\cos A=1-2\sin^2\dfrac{A}{2}$, we get

$$2\sin^2\frac{A}{2}=1-\cos A \text{ or } \sin\frac{A}{2}=\pm\sqrt{\frac{1-\cos A}{2}} \quad \ldots (1)$$

Also $\cos A=2\cos^2\dfrac{A}{2}-1$

$$\therefore\ 2\cos^2\frac{A}{2}=1+\cos A \text{ or } \cos\frac{A}{2}=\pm\sqrt{\frac{1+\cos A}{2}} \quad \ldots (2)$$

Dividing (1) by (2) we get

$$\tan\frac{A}{2}=\pm\sqrt{\frac{1-\cos\frac{A}{2}}{1+\cos\frac{A}{2}}} \quad \ldots \quad \ldots (3)$$

Note. The formulae (1), (2) and (3) have got ambiguous signs. To fix the proper sign the student should know in which quadrants $\frac{A}{2}$ lies. This following example illustrates the fact.

Example 1. *Find sin 165, cos 165° and tan 165°.*

Putting $A=330°$ in the above formulae

$$\sin 165° = \pm\sqrt{\frac{1-\cos 330°}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt3}{2}}{2}} = \pm\frac{\sqrt{2-\sqrt3}}{2}$$

$$\cos 165° = \pm\sqrt{\frac{1+\cos 330°}{2}} = \pm\sqrt{\frac{1+\frac{\sqrt3}{2}}{2}} = \pm\frac{\sqrt{2+\sqrt3}}{2}$$

$$\tan 165° = \pm\sqrt{\frac{2-\sqrt3}{2+\sqrt3}} = \pm\sqrt{\frac{(2-\sqrt3)^2}{4-3}} = \pm(2-\sqrt3)$$

Since 165° is angle in the second quadrant, its sine is positive, cosine and tangent negative.

Hence $\sin 165° = \dfrac{\sqrt{2-\sqrt3}}{2}$

$\cos 165° = -\dfrac{\sqrt{2+\sqrt2}}{2}$ and $\tan 165° = -(2-\sqrt3)$

8·71. To express

$\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ in terms of $\sin A$

We know that $\sin^2\frac{A}{2}+\cos^2\frac{A}{2}=1$

$$2\sin\frac{A}{2}\cos\frac{A}{2}=\sin A$$

$$\therefore\ \sin^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}+\cos^2\frac{A}{2}=1+\sin A$$

or $$\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2=1+\sin A$$

Hence $\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A}$ ......(1)

and similarly $\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A}$ ......(2)

Adding (1) and (2) and subtracting (2) from (1) we get

$$2\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}$$

$$2\cos\frac{A}{2}=\pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A}$$

Note 1 The above results should be remembered in the form

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A}$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A}$$

Note 2 The student should note it carefully that the signs to be given to the radicals not only depend upon the values of $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$ but on which of them is greater of the two. The student must decide the signs before addition or subtraction.

8 72. To decide the signs of the expressions

$\sin\frac{A}{2}+\cos\frac{A}{2}$ and $\sin\frac{A}{2}-\cos\frac{A}{2}$ for any given value of A.

Now $\sin\left(\frac{A}{2}+45°\right)=\sin\frac{A}{2}.\cos 45°+\cos\frac{A}{2}.\sin 45°$

$$=\frac{1}{\sqrt{2}}\sin\frac{A}{2}+\frac{1}{\sqrt{2}}\cos\frac{A}{2}$$

$$=\frac{\sin\frac{A}{2}+\cos\frac{A}{2}}{\sqrt{2}}$$

and $\sin\left(\frac{A}{2}-45°\right)=\sin\frac{A}{2}.\cos 45°-\cos\frac{A}{2}.\sin 45°$

$$=\frac{1}{\sqrt{2}}\sin\frac{A}{2}-\frac{1}{\sqrt{2}}.\cos\frac{A}{2}$$

$$=\frac{\sin\frac{A}{2}-\cos\frac{A}{2}}{\sqrt{2}}$$

In the above expressions $\sqrt{2}$ is taken positive, it being the values of sin 45° and cos 45°, which are both positive.

Hence $\sin \frac{A}{2}+\cos \frac{A}{2}$ and $\sin \frac{A}{2}-\cos \frac{A}{2}$ have the same signs as in $\sin\left(\frac{A}{2}+45°\right)$ and $\sin\left(\frac{A}{2}-45°\right)$

Example 1. *Given that $\sin 30°=\frac{1}{2}$, find $\sin 15°$ and $\cos 15°$.* (P. U.)

We know that

$$\sin 15°+\cos 15°=\pm\sqrt{1+\sin 30°}=\pm\sqrt{1+\tfrac{1}{2}}=\pm\frac{\sqrt{3}}{\sqrt{2}}$$

Now to decide the sign: The sign of sin 15°+cos 15° is the same as that of $\sin\left(\frac{A}{2}+45°\right)$, which $=\sin(15°+45°)$

$=\sin 60°=+$ve.

$$\therefore\ \sin 15°+\cos 15°=+\frac{\sqrt{3}}{\sqrt{2}} \qquad \ldots(1)$$

Again $\sin 15°-\cos 15°=\pm\sqrt{1-\sin 30°}=\pm\frac{1}{\sqrt{2}}$

To decide the sign: $\sin\left(\frac{A}{2}-45°\right)=\sin(15°-45°)=\sin(-30°)$, which is negative.

$$\therefore\ \sin 15°-\cos 15°=-\frac{1}{\sqrt{2}} \qquad \ldots(2)$$

$\therefore$ Adding (1) and (2), we get

$$2\sin 15°=\frac{\sqrt{3}-1}{\sqrt{2}} \text{ or } \sin 15°=\frac{\sqrt{3}-1}{2\sqrt{2}}$$

Subtracting (2) from (1)

$$2\cos 15°=\frac{\sqrt{3}+1}{\sqrt{2}} \text{ or } \cos 15°=\frac{\sqrt{3}+1}{2\sqrt{2}}$$

Example 2. *Given that $\sin 330°=-\frac{1}{2}$, find the values of sin 165 and cos 165°.*

We know that

$$\sin 165°+\cos 165°=\pm\sqrt{1+\sin 330°}=\pm\sqrt{1-\tfrac{1}{2}}=\pm\frac{1}{\sqrt{2}}$$

To decide the sign: $\sin\left(\frac{A}{2}+45°\right)=\sin(165°+45°)$ $=\sin(210°)=-\text{ve}.$

$\therefore\ \sin 165°+\cos 165°=-\frac{1}{\sqrt{2}}$ ...(1)

Again $\sin 165°-\cos 165°=\pm\sqrt{1-\sin 330°}=\pm\sqrt{2+\frac{1}{2}}$

$$=\pm\frac{\sqrt{3}}{\sqrt{2}}$$

To decide about the sign

$\sin\left(\frac{A}{2}-45°\right)=\sin(165°-45°)=\sin 120°=+\text{ve}$

$\therefore\ \sin 165°-\cos 165°=\frac{\sqrt{3}}{\sqrt{2}}$ ...(2)

From (1) and (2) by addition and subtraction, we get

$$\sin 165°=\frac{\sqrt{3}-1}{2\sqrt{2}} \text{ and } \cos 165°=-\frac{\sqrt{3}+1}{2\sqrt{2}}$$

8·73. To express $\tan\frac{A}{2}$ in terms of tan A.

We know that $\tan A=\dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

Putting $\tan\frac{A}{2}=t$, $\tan A=\frac{2t}{1-t^2}$ or $1-t^2=\frac{2t}{\tan A}$

$$\therefore\ t^2+\frac{2}{\tan A}.t-1=0$$

$$\therefore\ t=\frac{-\frac{2}{\tan A}\pm\sqrt{\frac{4}{\tan^2 A}+4}}{2}=\frac{-1\pm\sqrt{1+\tan^2 A}}{\tan A}$$

Example. *To find tan 75°, given that* $\tan 150°=-\frac{1}{\sqrt{3}}$

$$\tan 150°=\frac{2\tan 75°}{1-\tan^2 75°}=\frac{2t}{1-t^2} \text{ suppose}$$

$\therefore \quad (1-t^2)\left(-\frac{1}{\sqrt{3}}\right)=2t$ or $t^2-2\sqrt{3}t-1=0$

$\therefore \quad t=\frac{2\sqrt{3}\pm\sqrt{12+4}}{2}=\frac{2\sqrt{3}\pm 4}{2}=\sqrt{3}\pm 2$

But since tan 75° is positive the negative result is rejected.

$\therefore \quad t=\tan 75°=2+\sqrt{3}.$

EXAMPLES VIII (B)

1. If $\cos A=\frac{4}{5}$ and lies in the fourth quadrant, find $\sin\frac{A}{2}, \cos\frac{A}{2}, \tan\frac{A}{2}$.

2. Given $\sin A=\frac{\sqrt{3}}{2}$; find $\tan\frac{A}{2}$.

3. Given $\cos\theta=\frac{2mn}{m^2+n^2}$; find $\tan\frac{\theta}{2}$.

$$\left[\text{Apply } \tan^2\frac{\theta}{2}=\frac{1-\cos\theta}{1+\cos\theta}\right].$$

4. If $\cos 315°=\frac{1}{\sqrt{2}}$, find the value of cos 157° 30′.

5. Deduce the value of tan 15° from cos 30°. (*P.U.*)

[Hint : After putting down the values of cos 30°, rationalise the denominator.]

6. If A=420°, prove that

$$\sin\frac{A}{2}=\tfrac{1}{2}[-\sqrt{1+\sin A}+\sqrt{1-\sin A}]$$

$$\cos\frac{A}{2}=\tfrac{1}{2}[-\sqrt{1+\sin A}-\sqrt{1-\sin A}]$$

7 Attach proper signs to the radicals in the formulæ

(*i*) $2\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}$

(*ii*) $2\cos\frac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}$

when A=300°.

8. Determine the limits between which A must lie in the following case

(*i*) $2\cos\frac{A}{2}=+\sqrt{1+\sin A}-\sqrt{1-\sin A}$

(*ii*) $2\sin\frac{A}{2}=-\sqrt{1+\sin A}+\sqrt{1-\sin A}$

9. When $A=400°$, show that $\tan\frac{A}{2}=\frac{-1+\sqrt{1+\tan^2 A}}{\tan A}$

10. (*a*) Given $\tan 45°=1$, find $\tan 22\frac{1}{2}°$.

(*b*) Prove that $\tan 7\frac{1}{2}°=\sqrt{6}-\sqrt{3}+\sqrt{2}-2$.

3. FUNCTIONS OF 18° AND MULTIPLES OF 18°

8·8. To find the value of sin 18°.

Let $A=18°$ $\therefore$ $5A=90°$.

Hence $2A=90°-3A$

$\therefore$ $\sin 2A=\sin(90-3A)=\cos 3A$

or $2\sin A.\cos A=4\cos^3 A-3\cos A$

$\therefore$ Either $\cos A=0$ whence $A=90°$, which is rejected.

or $2\sin A=4(1-\sin^2 A)-3=1-4\sin^2 A$

$\therefore$ $4\sin^2 A+2\sin A-1=0$

$$\therefore \sin A=\frac{-2\pm\sqrt{4+16}}{8}=\frac{-2\pm2\sqrt{5}}{8}=\frac{-1\pm\sqrt{5}}{4}$$

But since sin 18° is necessarily positive

$\therefore$ $\sin 18°=\frac{\sqrt{5}-1}{4}$ and $\therefore$ $\cos 18°=\sqrt{1-\sin^2 18}$

$$=\sqrt{1-\frac{5+1-2\sqrt{5}}{16}}=\frac{\sqrt{10+2\sqrt{5}}}{4}$$

Cor. (*i*) $\cos 72°=\cos(90°-18°)=\sin 18°=\frac{\sqrt{5}-1}{4}$

Similarly (2) $\sin 72°=\frac{\sqrt{10+2\sqrt{5}}}{4}$

8·9. To find the value of cos 36°

Let $A=36°$ $\therefore$ $5A=180°$

Hence $2A=180°-3A$

$\therefore$ $\sin 2A=\sin(180°-3A)=\sin 3A$

or $2 \sin A \cdot \cos A = 3 \sin A - 4 \sin^3 A$

$\therefore$ Either $\sin A = 0$, *i.e.*, $A = 0$, which is rejected

or $2 \cos A = 3 - 4 \sin^2 A = 3 - 4(1 - \cos^2 A)$

$= 4 \cos^2 A - 1$

$\therefore \quad 4 \cos^2 A - 2 \cos A - 1 = 0$

or $\cos A = \dfrac{2 \pm \sqrt{4+16}}{8} \left[\because x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$

$= \dfrac{1 \pm \sqrt{5}}{4}$

$\cos 36°$ is necessarily +ve

$\therefore \quad \cos 36° = \dfrac{\sqrt{5}+1}{4}$

Now $\sin 36° = \sqrt{1 - \cos^2 36}$

$= \sqrt{1 - \left(\dfrac{\sqrt{5}+1}{4}\right)^2} = \dfrac{\sqrt{10 - 2\sqrt{5}}}{4}$

Cor. (*i*) $\sin 54° = \sin(90° - 36°) = \cos 36° = \dfrac{\sqrt{5}+1}{4}$

(*ii*) $\cos 54° = \cos(90° - 36°) = \sin 36°$

$= \dfrac{\sqrt{10 - 2\sqrt{5}}}{4}$

Note 1. The value of $\cos 36°$ can also be derived from the value of $\sin 18°$

We know that $\cos 2A = 1 - 2 \sin^2 A$

Putting $A = 18°$, we get

$\cos 36° = 1 - 2 \sin^2 18° = 1 - 2\left(\dfrac{\sqrt{5}-1}{4}\right)^2$

$= 1 - \dfrac{5+1-2\sqrt{5}}{8} = \dfrac{2+2\sqrt{5}}{8} = \dfrac{1+\sqrt{5}}{4}$

Hence $\cos 36° = \dfrac{\sqrt{5}+1}{4}$

Note 2. To deduce the value of $\sin 18°$ from the expansion of $\sin 5A$.

$\sin 5A = 5 \sin A - 20 \sin^3 A + 16 \sin^5 A$

Put $A = 18° \quad \therefore \sin 5A = \sin 90° = 1.$

Put $\sin A = x$

$\therefore\ 1 = 5x - 20x^4 + 16x^5$

or $16x^5 - 20x^3 + 5x - 1 = 0$

$16x^4(x-1) + 16x^3(x-1) - 4x^2(x-1) - 4x(x-1) + (x-1) = 0$

$(x-1)(16x^4 + 16x^3 - 4x^2 - 4x + 1) = 0$

$\therefore$ either $x - 1 = 0$ whence $\sin 18^\circ = 1$ which is rejected

$\because\ \sin 90^\circ = 1$

or $\qquad 16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$

i.e., $\qquad (4x^2 + 2x - 1)^2 = 0$

$\therefore \qquad 4x^2 + 2x - 1 = 0$

$$x = \frac{-1 \pm \sqrt{4+16}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

But since $\sin 18^\circ$ or x is necessarily positive

$$\therefore\quad \sin 18^\circ = \frac{\sqrt{5}-1}{4}.$$

Note 3. To deduce the value of $\cos 36^\circ$ from the expansion of $\cos 5A$.

$\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$

Put $A = 36$ and $\cos 36^\circ$ or $\cos A = x$

$\therefore\quad \cos 5A = \cos 180^\circ = -1$

$\therefore\quad -1 = 16x^5 - 20x^3 + 5x$

$16x^5 - 20x^3 + 5x + 1 = 0$

$16x^4(x+1) - 16x^3(x+1) - 4x^2(x+1) + 4x(x+1) + (x+1) = 0$

$\therefore\quad (x+1)(16x^4 - 16x^3 - 4x^2 + 4x + 1) = 0$

or $\qquad (x+1)(4x^2 - 2x - 1)^2 = 0$

$\therefore$ Either $x + 1 = 0$ whence $x = -1 = \cos 180^\circ$

i.e., $\qquad \cos A = \cos 180^\circ$

$\qquad A = 180^\circ$, which rejected

or $\qquad 4x^2 - 2x - 1 = 0$

$$\therefore\quad x = \frac{2 \pm \sqrt{4+16}}{8} = \frac{2 \pm 2\sqrt{5}}{8} = \frac{1 \pm \sqrt{5}}{4}$$

But $\cos 36^\circ$ is necessarily positive

$$\therefore\quad \cos 36^\circ = \frac{\sqrt{5}+1}{4}.$$

8·10. Another proof for the evaluation of $\sin 18^\circ$, $\cos 36^\circ$.

Denote $\cos 36^\circ$ and $\sin 18^\circ$ by x and y.

Now $2\sin 36^\circ \cos 36^\circ = \sin 72^\circ$

$2 \sin 72° \cos 72° = \sin 144° = \sin 36°$

$\therefore \quad 4 \sin 36° \sin 72° \cos 36° \cos 72° = \sin 72° \sin 36°$

$\therefore \quad 4 \cos 36° \cos 72° = 1$ *i.e.*, $4 \cos 36° \sin 18° = 1$

or $\quad 4xy = 1 \qquad \ldots\ldots(1)$

Also $x - y = \cos 36° - \cos 72° = 2 \sin 54° \sin 18° = 2xy$

i.e., $\quad x - y = 2xy \qquad \ldots\ldots(2)$

Eliminate x between (1) and (2)

$\therefore \quad x - \dfrac{1}{4x} = 2x \dfrac{1}{4x} = \dfrac{1}{2}$

$4x^2 - 2x - 1 = 0,$

which gives $x = \dfrac{\sqrt{5}+1}{4}$ (rejecting the inadmissible root)

From (1) $y = \dfrac{1}{4x} = \dfrac{1}{\sqrt{5}+1} = \dfrac{\sqrt{5}-1}{4}$

Hence $\sin 18° = \dfrac{\sqrt{5}-1}{4}$ and $\cos 36° = \dfrac{\sqrt{5}+1}{4}$

Example 1. *Show that* $\sec 72° - \sec 36° = 2$

$\sec 72° - \sec 36° = \dfrac{1}{\cos 72°} - \dfrac{1}{\cos 36°}$

$$= \frac{1}{\frac{\sqrt{5}-1}{4}} - \frac{1}{\frac{\sqrt{5}+1}{4}} = \frac{4}{\sqrt{5}-1} - \frac{4}{\sqrt{5}+1}$$

$$= \frac{4\{(\sqrt{5}+1) - (\sqrt{5}-1)\}}{5-1} = \frac{8}{4} = 2$$

Example 2. *To find the* $\sin 9°$ *and* $\cos 9°$

$\sin 9° = \sin (54° - 45°) = \sin 54° \cos 45° - \cos 54° \sin 45°$

$= \dfrac{\sqrt{5}+1}{4} \cdot \dfrac{1}{\sqrt{2}} - \dfrac{\sqrt{10-2\sqrt{5}}}{4} \cdot \dfrac{1}{\sqrt{2}}$

$= \dfrac{\sqrt{10}+\sqrt{2}}{8} - \dfrac{\sqrt{(5-\sqrt{5})}}{4}$

Similarly $\quad \cos 9° = \dfrac{\sqrt{10}+\sqrt{2}}{8} + \dfrac{\sqrt{(5-\sqrt{5})}}{4}$

Example 3. *Show that* $\sin 18°$ *and* $\sin 54°$ *are the roots of the equation* $\quad 4x^2 - 2\sqrt{5}x + 1 = 0$

Sum of the roots $= \sin 18° + \sin 54° = \sin 18° + \cos 36°$

$$= \frac{\sqrt{5}-1}{4} + \frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}}{2}$$

Product of the roots $= \sin 18^\circ . \sin 54^\circ$

$$= \frac{\sqrt{5}+1}{4} \times \frac{\sqrt{5}-1}{4} = \frac{1}{4}$$

Hence the equation whose roots are $\sin 18^\circ$ and $\sin 54^\circ$ is $x^2 - \frac{\sqrt{5}}{2}x + \frac{1}{4} = 0$ or $4x^2 - 2\sqrt{5}x + 1 = 0$

which is the given equation.

EXAMPLES VIII (C)

1. Given $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$, find the value of the following ratios

(*i*) $\sin 36^\circ$. (*ii*) $\sin 54^\circ$, (*iii*) $\tan 72^\circ$, (*iv*) $\cos 9^\circ$.

2. $\sin^2 72^\circ - \sin^2 60^\circ = \frac{\sqrt{5}-1}{8}$ (*D.U. 1951*)

3. $\sin 48^\circ \sin 12^\circ = \frac{\sqrt{5}-1}{8}$

4. $\sin 18^\circ + \sin 30^\circ = \sin 54^\circ$

5. Find (*i*) cosec 18° (*P.U. 1932*)

(*ii*) $\sec 36^\circ$ (*P.U. 1940-S*)

Show that

6. $\sin(36^\circ + A) + \sin(72^\circ - A) - \sin(36^\circ - A) - \sin(72^\circ - A) = \sin A$.

7. $\sin\frac{\pi}{10} . \sin\frac{13\pi}{10} = -\frac{1}{4}$.

8. $16 \cos\frac{2\pi}{15} . \cos\frac{4\pi}{15} . \cos\frac{8\pi}{15} . \cos\frac{14\pi}{15} = 1$ (*D.U. 1930*)

[**Hint** : Multiply and divide by $\sin\frac{2\pi}{15}$ and notice that $\cos\frac{14\pi}{15} = \cos\left(2\pi - \frac{14\pi}{15}\right) = \cos\frac{16\pi}{15}$]

9. $\sin 36^\circ . \sin 72^\circ . \sin 108^\circ . \sin 144^\circ = \frac{5}{16}$ (*P.U.*)

[**Hint** : L.H.S.

$$= \tfrac{1}{4}[4 \sin 36^\circ . \sin 72^\circ . \sin(180^\circ - 72^\circ) . \sin(180^\circ - 36^\circ)]$$
$$= \tfrac{1}{4}[4 \sin^2 36^\circ . \sin^2 72^\circ] = \tfrac{1}{4}[2 \sin 36^\circ . \sin 72^\circ]^2$$
$$= \tfrac{1}{4}[\cos(36^\circ) - \cos(108^\circ)]^2 = \tfrac{1}{4}[\cos 36^\circ + \sin 18^\circ]^2$$

10. $\cos 36^\circ . \cos 72^\circ . \cos 108^\circ . \cos 144^\circ = \frac{1}{16}$.

11. $\sin 6^\circ . \sin 42^\circ . \sin 66^\circ . \sin 78^\circ$
$= \cos 6^\circ . \cos 42^\circ . \cos 66^\circ . \cos 78^\circ = \frac{1}{16}$.

MISCELLANEOUS EXAMPLES ON CHAPTER VIII

Prove that (1 – 5)

1. (*i*) $\sin 4A = 4 \sin A \cos^3 A - 4 \cos A . \sin^3 A$

(*ii*) $\cos 4A = 8 \cos^4 A - 8 \cos^2 A + 1$.

2. (*i*) $4 (\cos^3 20^\circ + \cos^3 40^\circ) = 3 (\cos 20^\circ + \cos 40^\circ)$

(*ii*) $\dfrac{\sin 3\alpha + \sin^3 \alpha}{\cos^3 \alpha - \cos 3\alpha} = \cot \alpha$.

3. (*i*) $\cot 2A = \dfrac{\cot^2 A - 1}{2 \cot A}$ (*P.U.*)

(*ii*) $\dfrac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} = 4 \cos 2\theta . \cos 4\theta$ (*P.U.*)

(*iii*) $\dfrac{1}{\tan 3A - \tan A} - \dfrac{1}{\cot 3A - \cot A} = \cot 2A$

(*P.U. 1944-S*)

[**Hint :** Change into sines and cosines.]

4. $\dfrac{\sec 8\theta - 1}{\sec 4\theta - 1} = \dfrac{\tan 8\theta}{\tan 2\theta}$ (*P.U.*)

[**Hint :** L.H.S. =

$$\frac{1 - \cos 8\theta}{\cos 8\theta} . \frac{\cos 4\theta}{1 - \cos 4\theta} = \frac{2 \sin^2 4\theta}{\cos 8\theta} . \frac{\cos 4\theta}{2 \sin^2 2\theta}$$

$$= \frac{2 \sin 4\theta . \cos 4\theta \sin 4\theta}{\cos 8\theta . 2 \sin^2 2\theta} = \frac{\sin 8\theta . 2 \sin 2\theta . \cos 2\theta}{\cos 8\theta . 2 \sin^2 2\theta}]$$

5. $2 \cos \theta = \sqrt{2 + \sqrt{2 + 2 \cos 4\theta}}$ (*P.U. 1916*)

[**Hint :** R.H.S. $= \sqrt{2 + \sqrt{2(1 + \cos 4\theta)}} = \sqrt{2 + \sqrt{4 \cos^2 2\theta}}$

$= \sqrt{2 + 2 \cos 2\theta}$]

6. If $\cos \theta = \frac{1}{2}\left(a + \dfrac{1}{a}\right)$, show that (*i*) $\cos 2\theta = \frac{1}{2}\left(a^2 + \dfrac{1}{a^2}\right)$

(*ii*) $\cos 3\theta = \frac{1}{2}\left(a^3 + \dfrac{1}{a^3}\right)$ (*P.U. 1915*)

7. If $x^2+y^2=1$, prove that $(3x-4x^3)^2+(3y-4y^3)^2=1$. (*P.U.*)

[Put $x=\sin\alpha$

$y=\cos\alpha$, then L.H.S. $=(\sin 3\alpha)^2+(-\cos 3\alpha)^2$].

8. (*i*) If $\cot A=\frac{a}{b}$ find $\sin 2A$. (*P.U.*)

(*ii*) If $\tan\alpha=1+\sqrt{2}$, find $\cos 2\alpha$. (*P.U.*)

9. (*a*) Prove that $\cos^2 A+\cos^2(60^\circ+A)+\cos^2(60^\circ-A)=\frac{3}{2}$

(*b*) Show that $(2\cos A-1)(2\cos 2A-1)(2\cos 2^2A-1)......$

$$(2\cos 2^{n-1}A-1)=\frac{2\cos 2^nA-1}{2\cos A+1}$$

10. Prove that

(*i*) $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)=\frac{1}{8}$

(*ii*) $\cos^2\frac{\pi}{8}+\cos^2\frac{3\pi}{8}+\cos^2\frac{5\pi}{8}+\cos^2\frac{7\pi}{8}=2$

(*iii*) $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\cos^4\frac{5\pi}{8}+\cos^4\frac{7\pi}{8}=\frac{3}{2}$

[Hint : (*i*) $\cos\frac{7\pi}{8}=-\cos\frac{\pi}{8}$ and $\cos\frac{5\pi}{8}=-\cos\frac{3\pi}{8}$

$\therefore$ Combining 1st and 4th factors and 2nd and 3rd factors we get

$$\text{Expr.}=\left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right)=\sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}$$

$$=\sin^2\frac{\pi}{8}\cdot\cos^2\frac{\pi}{8}\left[\because \sin\frac{3\pi}{8}=\cos\frac{\pi}{8}\right]=\frac{1}{4}\cdot\sin^2\frac{\pi}{4}=\frac{1}{8}$$

(*iii*) $\cos^4\frac{7\pi}{8}=\cos^4\frac{\pi}{8}$ and $\cos^4\frac{5\pi}{8}=\cos^4\frac{3\pi}{8}$

Also $\cos^4\frac{\pi}{8}=\left(\frac{1+\cos\frac{\pi}{4}}{2}\right)^2$ and $\cos^4\frac{3\pi}{8}=\left(\frac{1+\cos\frac{3\pi}{4}}{2}\right)^2$]

11. Find $\sin 9^\circ$ and $\cos 9^\circ$, taking $\sin 18^\circ=\frac{\sqrt{5}-1}{4}$ (*P.U.*)

[$\sin 9^\circ + \cos 9^\circ = \pm\sqrt{1+\sin 18^\circ} = \pm\sqrt{1+\dfrac{\sqrt{5}-1}{4}}$

$$=\frac{\sqrt{3+\sqrt{5}}}{2}$$

Now $\sin\left(\dfrac{A}{2}+45^\circ\right)=\sin(9^\circ+45^\circ)=\sin 54^\circ=+$ve.

$\therefore\quad \sin 9^\circ+\cos 9^\circ=+\dfrac{\sqrt{3+\sqrt{5}}}{2}$ ...(1)

and $\sin 9^\circ-\cos 9^\circ=\pm\sqrt{1-\sin 18^\circ}=\pm\dfrac{\sqrt{5-\sqrt{5}}}{2}$

Now $\sin\left(\dfrac{A}{2}-45^\circ\right)=\sin(9^\circ-45)=\sin(-36^\circ)=-$ ve

$\therefore\quad \sin 9^\circ-\cos 9^\circ=-\dfrac{\sqrt{5-\sqrt{5}}}{2}$ ...(2)

From (1) and (2) we can find $\sin 9^\circ$ and $\cos 9^\circ$.]

12. In the formula $2\sin\dfrac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}$ determine the proper signs of the radicals if

A lies in the 3rd quadrant :

[A lies between 180° and 270° $\therefore$ $\dfrac{A}{2}$ lies between 90° and 135° $\therefore$ $\sin\left(\dfrac{A}{2}+45^\circ\right)$ lies between $\sin(135^\circ)$ and $\sin(180^\circ)$ $\therefore$ $\sin\left(\dfrac{A}{2}+45^\circ\right)$ is $+$ve; Again subtracting 45° from $\dfrac{A}{2}$, we find that $\sin\left(\dfrac{A}{2}-45^\circ\right)$ lies between $\sin 45^\circ$ and $\sin 90^\circ$ $\therefore$ $\sin\left(\dfrac{A}{2}-45^\circ\right)$ is $+$ve.]

13. If $\cos\theta=\dfrac{\cos u-e}{1-e\cos u}$, prove that

$$\tan\frac{\theta}{2}=\pm\sqrt{\frac{1+e}{1-e}}\cdot\tan\frac{u}{2}$$ *(D. U.)*

[By Applying Div. and Comp: we get

$$\frac{1-\cos\theta}{1+\cos\theta}=\frac{(1-\cos u)+e(1-\cos u)}{(1+\cos u)-e(1+\cos u)}=\frac{(1+e)(1-\cos u)}{(1-e)(1+\cos u)}$$

$$\therefore\ \tan^2\frac{\theta}{2}=\frac{1+e}{1-e}\cdot\tan^2\frac{u}{2}.]$$

14. Prove that $\tan\alpha+2\tan 2\alpha+4\tan 4\alpha+8\cot 8\alpha=\cot\alpha$.
(*P. U. 1935*)

15. Given $\tan\alpha=\dfrac{1+\sqrt{1-a}}{1+\sqrt{1+a}}$, show that $\sin 4\alpha=a$.

(*P. U. 1930*)

[Put $a=\sin 2\theta$, we get

$$\tan\alpha=\frac{1+\sqrt{1-\sin 2\theta}}{1+\sqrt{1+\sin 2\theta}}=\frac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}$$

$$=\frac{(1-\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta}=\tan\frac{\theta}{2},$$

$$\therefore\quad \alpha=\frac{\theta}{2}\quad\text{or}\quad 4\alpha=2\theta\quad\text{or}\quad \sin 4\alpha=\sin 2\theta=a.]$$

16. If $\sin\theta=\dfrac{m^2-n^2}{m^2+n^2}$, find $\tan\dfrac{\theta}{2}$.

17. Prove that $\cos 12^\circ+\cos 60^\circ+\cos 84^\circ$
$=\cos 24^\circ+\cos 48^\circ$. (*P. U.*)

18. Two parallel chords of a circle, which are on the same side of the centre subtend angles of 72° and 144° respectively at the centre. Prove that the perpendicular distance between the chords is half the radius. (*P. U. 1939-S*)

19. In any circle prove that the chord which subtends 108° at the centre is equal to the sum of two chords which subtend angles of 36° and 60°.

20. Show that $2\sin 18^\circ$ and $2\cos 144^\circ$ are the roots of the equation $x^2+x-1=0$.

ANSWERS TO EXERCISES IN CHAPTER VIII

Exercises VIII (A)

1. $\dfrac{120}{169}$, $-\dfrac{119}{169}$. 2. $\dfrac{3}{5}$, $\dfrac{4}{5}$, $\dfrac{3}{4}$; 4. a. 5. $\dfrac{13}{9}$

6. $\dfrac{117}{125}$, $\dfrac{44}{125}$.

Exercises VIII (B)

1. $\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}, -\frac{1}{3}$; 2. $\sqrt{3}$ or $\frac{1}{\sqrt{3}}$. 3. $\pm\frac{m-n}{m+n}$.

4. $-\frac{\sqrt{2+\sqrt{2}}}{2}$ 5. $2-\sqrt{3}$.

7. $2 \sin \frac{A}{2} = -\sqrt{1+\sin A} + \sqrt{1-\sin A}$; $2 \cos \frac{A}{2}$

$= -\sqrt{1+\sin A} - \sqrt{1-\sin A}$;

8. (*i*) $90°$ and $270°$. (*i*) $270°$ and $450°$. 10. $\sqrt{2}-1$.

Exercises VIII (C)

1. (*i*) $\frac{1}{4}(10-2\sqrt{5})$. (*ii*) $\frac{\sqrt{5}+1}{4}$.

(*iii*) $\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$. (*iv*) 0.

5. (*i*) $\frac{4}{\sqrt{5}-1}$ (*ii*) $-\frac{4}{\sqrt{5}+1}$.

Miscellaneous Examples on Chapter VIII

7. (*i*) $\frac{2ab}{a^2+b^2}$. (*ii*) $\pm\frac{1}{\sqrt{2}}$.

14. [Hint: $\cot a - \tan a = \cot a - \frac{1}{\cot a} = \frac{2(\cot^2 a - 1)}{2 \cot a} = 2 \cot 2a$

$\therefore (\cot a - \tan a) - 2 \tan 2a - 4 \tan 4a$

$= 2(\cot 2a - \tan 2a) - 4 \tan 4a$

$= 4(\cot 4a - \tan 4a) = 8 \cot 8a$.

CHAPTER IX

TRIGONOMETRIC IDENTITIES AND TRIGONOMETRICAL EQUATIONS

SECTION 1

9·1. Trigonometrical Identities when $A+B+C=180°$

When there are three angles A, B, C such that their sum is 180° (or they form the angles of a triangle) many identities are found to hold between the trigonometrical ratios or the trigonometrical ratios of their multiple angles.

Some Typical Examples shall best illustrate the subject.

Example 1. *If $A+B+C=180°$, prove that*
$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\cdot\sin C$. (*C. U. 1920*)

$$\text{L. H. S.}=(\sin 2A+\sin 2B)+\sin 2C.$$

(Express the sum of the first two as a product and form factors of the third term)

$$=2\sin(A+B)\cdot\cos(A-B)+2\sin C\cdot\cos C.$$

(Now we know the result is expressed in the form of product of three factors. Hence either sin A, sin B or sin C must be a common factor of the expression on the R. H. S. in the second step.) We find that

$$\sin(A+B)=\sin(180-C)=\sin C$$

$$\therefore\quad \text{Expr.}=2\sin C\cos(A-B)+2\sin C\cos C$$
$$=2\sin C.[\cos(A-B)+\cos C]$$

(when we have arrived at one common factor in 'C', we must realise that the other terms must contain only A and B.

$\therefore$ we express cos C in terms of A and B. Hence

$$\cos C=\cos\{180°-\overline{A+B}\}=-\cos(A+B)$$

$$\therefore\quad \text{Expr.}=2\sin C.[\cos(A-B)-\cos(A+B)]$$
$$=2\sin C.2\sin A.\sin B$$
$$=4\sin A.\sin B\sin C.$$

Note. The student should note that explanation given within brackets may not be given by him while solving these identities. He should, however, have these steps very carefully in mind.

Example 2. *If $A+B+C=180^\circ$ prove that*

$$\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \qquad (P.U.\ 1912)$$

$$(\cos A+\cos B)+\cos C=2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\left(1-2\sin^2\frac{C}{2}\right)$$

But $\cos\frac{A+B}{2}=\cos\left(90^\circ-\frac{C}{2}\right)=\sin\frac{C}{2}$.

$$\therefore \quad \text{L. H. S.}=2\sin\frac{C}{2}\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}+1$$

$$=2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)+1$$

(taking $\sin\frac{C}{2}$ as a common factor)

$$=2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right)+1$$

(expressing the terms within brackets in A and B)

$$=2\sin\frac{C}{2}\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)+1$$

$$=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

Example 3. *If $A+B+C=180^\circ$ prove that*

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$$
(P.U. 1919, 1943

Now $\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C$

$$=\left(\frac{1+\cos 2A}{2}\right)+\left(\frac{1+\cos 2B}{2}\right)+\cos^2 C+2\cos A\cos B\cos C$$

$$=1+\tfrac{1}{2}(\cos 2A+\cos 2B)+\cos^2 C+2\cos A\cos B\cos C$$

$$=1+\tfrac{1}{2}\{2\cos(A+B)\cos(A-B)\}+\cos^2 C+2\cos A\cos B\cos C$$

[Now $\cos(A+B)=\cos(180^\circ-C)=-\cos C$]

$=1+\cos(180-C)\cos(A-B)+\cos^2 C+2\cos A\cos B\cos C$

$=1-\cos C\cos(A-B)+\cos^2 C+2\cos A\cos B\cos C$

$=1-\cos C\{\cos(A-B)-\cos C\}+2\cos A\cos B\cos C$

$=1-\cos C\{\cos(A-B)-\cos(180^\circ-\overline{A+B})\}+$
$2\cos A\cos B\cos C$

$=1-\cos C\{\cos(A-B)+\cos(A+B)\}+2\cos A\cos B\cos C$

[Note this step: The terms within bracket have been expressed in A and B.]

$=1-\cos C\{2\cos A\cos B\}+2\cos A\cos B\cos C$

$=1-2\cos A\cos B\cos C+2\cos A\cos B\cos C=1.$

Example 4. *If $A+B+C=180^\circ$, prove that*

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=1+4\sin\frac{\pi-A}{4}.\sin\frac{\pi-B}{4}.\sin\frac{\pi-C}{4}.$$

Now $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$

$=\cos\left(\frac{\pi}{2}-\frac{A}{2}\right)+\cos\left(\frac{\pi}{2}-\frac{B}{2}\right)+\cos\left(\frac{\pi}{2}-\frac{C}{2}\right)$

$=\cos X+\cos Y+\cos Z$

where $X+Y+Z=\frac{\pi}{2}-\frac{A}{2}+\frac{\pi}{2}-\frac{B}{2}+\frac{\pi}{2}-\frac{C}{2}$

$=\frac{3\pi}{2}-\frac{1}{2}(A+B+C)=\frac{3\pi}{2}-\frac{\pi}{2}=\pi.$

Since $X+Y+Z=180^\circ$

$\therefore$ by example 2 or working directly

$$\cos X+\cos Y+\cos Z=1+4\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}$$

$$=1+4\sin\frac{\pi-A}{4}.\sin\frac{\pi-B}{4}.\sin\frac{\pi-C}{4}.$$

$\therefore\quad \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$

$$=1+4\sin\frac{\pi-A}{4}.\sin\frac{\pi-B}{4}.\sin\frac{\pi-C}{4}$$

OR Second Method.

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}$$

$$=2\sin\frac{A+B}{4}.\cos\frac{A-B}{4}+\cos\left(\frac{\pi-C}{2}\right) \quad \text{(Note the step)}$$

$$=2\sin\frac{\pi-C}{4}.\cos\frac{A-B}{4}+\left(1-2\sin^2\frac{\pi-C}{4}\right)$$

$$=1+2\sin\frac{\pi-C}{4}\cos\frac{A-B}{4}-2\sin^2\frac{\pi-C}{4}$$

$$=1+2\sin\frac{\pi-C}{4}\cdot\left[\cos\frac{A-B}{4}-\sin\frac{\pi-C}{4}\right]$$

$$=1+2\sin\frac{\pi-C}{4}\cdot\left[\cos\frac{A-B}{4}-\sin\frac{A+B}{4}\right]$$

(Changing inside terms into A and B)

$$=1+2\sin\frac{\pi-C}{4}\cdot\left[\cos\frac{A-B}{4}-\cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\right]$$

(note this step.)

$$=1+2\sin\frac{\pi-C}{4}.\ 2\sin\frac{\pi-A}{4}.\sin\frac{\pi-B}{4}$$

$$=1+4\sin\frac{\pi-A}{4}.\sin\frac{\pi-B}{4}.\sin\frac{\pi-C}{4}.$$

Example 5. *Show that*

$\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)$
$=4\sin A.\sin B.\sin C.$ (P. U. 1922)

L. H. S. $=\sin(A+B+C-2A)+\sin(C+A+B-2B)$
$+\sin(A+B+C-2C)$
$=\sin(180^\circ-2A)+\sin(180^\circ-2B)+\sin(180^\circ-2C)$
$=\sin 2A+\sin 2B+\sin 2C.$

which by virtue of (1) solved example
$=4\sin A.\sin B.\sin C.$

Example 6. *If $A+B+C=180^\circ$ prove that*

(i) *$\tan A+\tan B+\tan C=\tan A\tan B\tan C.$*
(P. U. 1938, 46)

(ii) *$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$*
(P. U. 1918, 1941, 1942)

(*i*) Since $A+B+C=180°$.

$\therefore \quad \tan(A+B)=\tan(180°-C)=-\tan C.$

$\therefore \quad \dfrac{\tan A+\tan B}{1-\tan A \tan B}=-\tan C.$

or $\quad \tan A+\tan B=-\tan C+\tan A \tan B \tan C.$

Hence $\tan A+\tan B+\tan C=\tan A \tan B \tan C.$

(*ii*) Since $A+B+C=180°$

$\therefore \quad \cot\left(\dfrac{A}{2}+\dfrac{B}{2}\right)=\cot\left(90°-\dfrac{C}{2}\right)=\tan\dfrac{C}{2}$

$\therefore \quad \dfrac{\cot\dfrac{A}{2}\cot\dfrac{B}{2}-1}{\cot\dfrac{A}{2}+\cot\dfrac{B}{2}}=\dfrac{1}{\cot\dfrac{C}{2}}$

$\therefore \quad \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}-\cot\dfrac{C}{2}=\cot\dfrac{A}{2}+\cot\dfrac{B}{2}$

Hence $\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}=\cot\dfrac{A}{2}+\cot\dfrac{B}{2}+\cot\dfrac{C}{2}$

Another Method.

(*i*) $A+B+C=180° \quad \therefore \tan(A+B+C)=\tan 180°$

or $\quad \dfrac{\tan A+\tan B+\tan C-\tan A \tan B \tan C}{1-\tan B \tan C-\tan C \tan A-\tan A \tan B}=0$

Hence $\tan A+\tan B+\tan C=\tan A.\tan B.\tan C.$

(*ii*) $A+B+C=180° \quad \therefore \dfrac{A+B+C}{2}=90°$

$\therefore \quad \tan\left(\dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}\right)=\tan 90°$

or $\quad \dfrac{\tan\dfrac{A}{2}+\tan\dfrac{B}{2}+\tan\dfrac{C}{2}-\tan\dfrac{A}{2}.\tan\dfrac{B}{2}.\tan\dfrac{C}{2}}{1-\tan\dfrac{A}{2}.\tan\dfrac{B}{2}-\tan\dfrac{B}{2}.\tan\dfrac{C}{2}-\tan\dfrac{C}{2}.\tan\dfrac{A}{2}}=\infty.$

$\therefore$ The denominator is zero

or $\quad 1=\tan\dfrac{A}{2}.\tan\dfrac{B}{2}+\tan\dfrac{B}{2}.\tan\dfrac{C}{2}+\tan\dfrac{C}{2}.\tan\dfrac{A}{2}$

Multiply by $\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}$ we get

$$\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$$

Example 7. *If $x+y+z=xyz$, prove that*

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}$$

Put $x=\tan A$, $y=\tan B$, $z=\tan C$

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C$$

$$\therefore A+B+C=180°$$

Now $\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2z}{1-z^2}$

$$=\frac{2\tan A}{1-\tan^2A}+\frac{2\tan B}{1-\tan^2B}+\frac{2\tan C}{1-\tan^2C}$$

$$=\tan 2A+\tan 2B+\tan 2C$$

$$=\tan 2A.\tan 2B.\tan 2C \qquad \text{[Ex. 6]}$$

$$=\frac{2\tan A}{1-\tan^2A}\cdot\frac{2\tan B}{1-\tan^2B}\cdot\frac{2\tan C}{1-\tan^2C}$$

$$=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2z}{1-z^2}.$$

EXAMPLES IX (A)

If $A+B+C=180°$, prove that

1. $\sin 2A-\sin 2B+\sin 2C=4\cos A\sin B\cos C.$
2. $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$ *(P. U. 1917, 45, 47-S)*
3. $\cos A-\cos B+\cos C+1=4\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}.$
4. $\sin A+\sin B-\sin C=4\sin\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2}$
5. $\sin 4A+\sin 4B+\sin 4C=-4\sin 2A.\sin 2B\sin 2C$ *(P. U. 1917)*
6. $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A.\cos B.\cos C$ *(P. U.)*

7. $\cos 2A+\cos 2B-\cos 2C=1-4 \sin A.\sin B \sin C.$ *(P. U. 1942-S)*

(Exercise 8–10 are of the type of 3rd solved example)

8. $\cos^2 A+\cos^2 B-\cos^2 C=1-2 \sin A.\sin B.\cos C.$

9. (*i*) $\sin^2 A+\sin^2 B+\sin^2 C \quad 2(1+\cos A.\cos B.\cos C)$ *(D. U. 1939)*

(*ii*) $\sin^2 A+\sin^2 B-\sin^2 C=2 \sin A.\sin B.\cos C.$

10. (*i*) $\sin^2 \frac{A}{2}+\sin^2 \frac{B}{2}+\sin^2 \frac{C}{2}=1-2 \sin \frac{A}{2}.\sin \frac{B}{2}.\sin \frac{C}{2}$

(*ii*) $\sin^2 \frac{A}{2}+\sin^2 \frac{B}{2}-\sin^2 \frac{C}{2}=1-2 \cos \frac{A}{2}.\cos \frac{B}{2}.\cos \frac{C}{2}$

11. $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8 \sin \frac{A}{2}.\sin \frac{B}{2}.\sin \frac{C}{2}$

12. $\dfrac{\sin B+\sin C-\sin A}{\sin B+\sin C+\sin A}=\tan \frac{B}{2}.\tan \frac{C}{2}$ *(D. U. 1937)*

13. (*i*) $\cos \frac{A}{2}+\cos \frac{B}{2}+\cos \frac{C}{2}=4 \cos \frac{\pi-A}{4}$

$\cos \frac{\pi-B}{4}.\cos \frac{\pi-C}{4}.$ *(D. U. 1937)*

[**Hints :** Proceed as in solved Example 4]

(*ii*) $\cos \frac{A}{2}+\cos \frac{B}{2}+\cos \frac{C}{2}=4 \cos \frac{B+C}{4}.\cos \frac{C+A}{4}.\cos \frac{A+B}{4}$

14. If A, B, C are the angles of a triangle, prove that

$\tan \frac{B}{2}.\tan \frac{C}{2}+\tan \frac{C}{2}.\tan \frac{A}{2}+\tan \frac{A}{2}.\tan \frac{B}{2}=1.$

[**Hint :** Proceed in solved Example (6)]

15. If $A+B+C=180°$, prove that

$\cot B.\cot C+\cot C.\cot A+\cot A.\cot B=1$

[Proceed in solved Example (6) OR

$\cot (A+B)=\cot (180°-C)$

or $\dfrac{\cot A.\cot B-1}{\cot A+\cot B}=-\cot C$

Cross-multiply and transpose]

16. Prove that $\tan (B-C)+\tan (C-A)+\tan (A-B)$

$=\tan (B-C) \tan (C-A).\tan (A-B)$ *(P. U. 1935)*

$B-C=x,\ C-A=y,\ A-B=z$

$$x+y+z=0$$

∴ By proceeding as in solved Example (6), we get

$$\tan x+\tan y+\tan z=\tan x \tan y \tan z$$

or $\tan(B-C)+\tan(C-A)+\tan(A-B)=\tan(B-C)\cdot\tan(C-A)\tan(A-B)$

II. TRIGONOMETRIC EQUATIONS

(a) Simple Equations

9·2. The various methods that can be used in solving the equations will be illustrated by examples. We shall for the present restrict ourselves to those values of the unknown variable which lie between 0° and 360°.

Example 2. *Solve the equation for all solution from 0° to 360°, $2\cos^2\theta=1+\sin\theta$.*

The first step in the procedure is to express both sides of the equation in terms of only one trigonometrical ratio.

since $\cos^2\theta=1-\sin^2\theta$

∴ $2(1-\sin^2\theta)=1+\sin\theta$

∴ $2-2\sin^2\theta=1+\sin\theta$

∴ $2\sin^2\theta+\sin\theta-1=0$

∴ $(2\sin\theta-1)(\sin\theta+1)=0$

∴ Either $\sin\theta=\frac{1}{2}$ or -1

If $\sin\theta=\frac{1}{2}$, θ may be 30° or 150°

If $\sin\theta=-1$, θ is 270°

Hence $\theta=30°, 150°, 270°$

Example 2. *Solve the equation $2\cos^2\theta=3\sin\theta$*

∴ $2(1-\sin^2\theta)=3\sin\theta$

or $2-2\sin^2\theta=3\sin\theta$

∴ $2\sin^2\theta+3\sin\theta-2=0$

or $(2\sin\theta-1)(\sin\theta+2)=0$

∴ $\sin\theta=\frac{1}{2}$ or -2

The value -2 is rejected being impossible.

∴ $\sin\theta=\frac{1}{2}$ when $\theta=30°$ or 150°

Example 3. *Solve the equation $\sin 2\theta\cos\theta=\sin 3\theta\cos 2\theta$.*

∴ $\sin 2\theta\cos\theta=\sin 3\theta\cos 2\theta$

∴ $2\sin 2\theta\cos\theta=2\sin 3\theta\cos 2\theta$

or $\sin(2\theta+\theta)+\sin(2\theta-\theta)=\sin(3\theta+2\theta)+\sin(3\theta-2\theta)$

i.e., $\sin 3\theta+\sin\theta=\sin 5\theta+\sin\theta$

$\therefore$ $\sin 3\theta=\sin 5\theta$

$\therefore$ $5\theta=3\theta$ or $180-3\theta$

i.e, either $\theta=0$ or $8\theta=180°$, $\theta=22\frac{1}{2}°$

Example 4. *Solve $\sin 2\theta=\cos 3\theta$ for values between 0° and 360°*

$\therefore$ $2\sin\theta\cos\theta=4\cos^3\theta-3\cos\theta$

$\therefore$ Either $\cos\theta=0$ whence $\theta=\pm 90°$

or $2\sin\theta=4\cos^2\theta-3=4(1-\sin^2\theta)-3$

$\therefore$ $4\sin^2\theta+2\sin\theta-1=0$

$$\therefore\quad \sin\theta=\frac{-2\pm\sqrt{4+16}}{8}=\frac{-2\pm 2\sqrt{5}}{8}=\frac{-1\pm\sqrt{5}}{4}$$

$$=\frac{\sqrt{5}-1}{4}\text{ or }-\left(\frac{\sqrt{5}+1}{4}\right)$$

$\therefore$ $\theta=18°$ or $-54°$

$\therefore$ $\theta=18°, 90°, -54°, -90°$

or $\theta=18°, 90°, 306°$ and $270°$

Example 5. *Solve for values from 0 to 180°*

$$3\sin\theta+4\cos\theta=5$$

1st Method. Divide by $\sqrt{3^2+4^2}$ *i.e.*, 5 both sides of the equation.

$\therefore$ $\frac{3}{5}\sin\theta+\frac{4}{5}\cos\theta=1$

Construct an angle α, so that

$\tan\alpha=\frac{4}{3}$

$\therefore$ $\sin\theta\cos\alpha+\cos\theta\sin\alpha=1$

or $\sin(\theta+\alpha)=1$

$\therefore$ $\theta+\alpha=90$ $\qquad\therefore$ $\theta=90-\alpha$

Now from tables $\tan\alpha=\frac{4}{3}$

or $\cot\alpha=\frac{3}{4}=\cdot75$ gives

$\alpha=53°8'$

$\therefore$ $\theta=36°52'$

2nd Method. $3\sin\theta+4\cos\theta=5$

$$\therefore\quad 3\,\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}+4\,\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=5$$

Putting $\tan\dfrac{\theta}{2}=x$

$$6x+4(1-x^2)=5(1+x^2)$$

$$\therefore \quad 9x^2-6x+1=0$$

$$(3x-1)^2=0 \quad \therefore \quad x=\tfrac{1}{3}$$

$$\therefore \quad \tan\frac{\theta}{2}=\frac{1}{3}=\cdot 3333$$

From the tables $\dfrac{\theta}{2}=18°26'$

$$\therefore \qquad \theta=36°52'$$

Note. The method for consulting the tables is discussed in Chapter XI.

Example 6. *Solve approximately the equation* $\sin\theta=\cdot 51$ *(P. U. 1939)*

$\sin\theta$ is very nearly equal to $\frac{1}{2}$, θ is nearly equal to $30°$

Let $\theta=30°+x$

$$\therefore \quad \cdot 51=\sin(\theta)=\sin(30°+x)=\sin 30°\cos x+\cos 30°\sin x$$

$$=\tfrac{1}{2}\cos x+\frac{\sqrt{3}}{2}\sin x$$

Since x is very small, we can take

$\cos x=1$ and $\sin x=x$

[We shall prove in the next chapters that $\text{Lt}\dfrac{\sin x}{x}=1$, when $x\to 0$]

$$\therefore \quad \cdot 51=\tfrac{1}{2}+\frac{\sqrt{3}}{2}x$$

$$\therefore \quad x=\cdot 01\times\frac{2}{\sqrt{3}}\text{ radians}=\frac{1}{100}\times\frac{2\sqrt{3}}{3}\text{ radians.}$$

$$=39'\,40''\text{ nearly}$$

$$\therefore \quad \theta=30°\,39'\,40'\text{ nearly.}$$

EXERCISES IX (B)

Solve the following equations, giving all solutions from 0 to 360°.

1. $\sin^2\theta+2\sin\theta=2-\cos^2\theta$.
2. $\tan\theta=2\sin\theta$.
3. $\sin\theta-\cos\theta=\sqrt{2}$.
4. $\cos 4\theta-\cos 2\theta=0$.
5. $\sin 4\theta\cos\theta=\sin 3\theta\cos 2\theta$.

6. $\sin 2\theta = 1 + \cos 2\theta$.

7. $\cos \theta + \cos 3\theta = \cos 2\theta$. (P. U. 1940)

8. $\cot 2\theta - \cot \theta = \frac{2}{\sqrt{3}}$.

Solve approximately the equations :—

9. $\sin \theta = \cdot 52$. 10. $\cos \theta = \cdot 49$.

(b) General equations

93. We know that if an angle is given, its sine or cosine has a definite value. The converse statement is not true, for angle whose sines or cosines are equal to the given value.

For example, consider the angles

$$30^\circ, 150^\circ, 390^\circ, 510^\circ, 750^\circ, 870^\circ \ldots\ldots.$$

$$\sin 30^\circ = \tfrac{1}{2}$$
$$\sin 150^\circ = \sin (180^\circ - 30^\circ) = \sin 30^\circ = \tfrac{1}{2}$$
$$\sin 390^\circ = \sin (360^\circ + 30^\circ) = \sin 30^\circ = \tfrac{1}{2}$$
$$\sin 510^\circ = \sin (360^\circ + 150^\circ) = \sin 150^\circ = \tfrac{1}{2}$$
$$\sin 750^\circ = \sin (360^\circ + 390^\circ) = \tfrac{1}{2}$$
$$\sin 870^\circ = \sin (360^\circ + 510^\circ) = \sin 510^\circ = \tfrac{1}{2}$$

Hence if $\sin \theta = \frac{1}{2}$ be given, we cannot definitely say that $\theta = 30^\circ$ or 150° etc., but we can say that θ has one of the values $30^\circ, 150^\circ, 390^\circ, 510^\circ$......etc.

Of these values 30° is the least positive value.

It is now our aim to find a single expression to denote all the angles whose sine is a given number, say $\frac{1}{2}$.

11. **Coterminal angles.** If OX be the initial line and OX any revolving line which traces out an angle XOP, then all angles which are formed by the same position of OP, in different revolutions either in a clockwise or anti-clockwise direction are said to be coterminal.

In the above article

$30^\circ, 390^\circ, 750^\circ$......are coterminal angles and so are $150^\circ, 510^\circ, 870^\circ$.....

In the above fig. if $\angle XOP$ has the least positive value α, then $\angle XOP$ represents not only $\angle \alpha$ but all angles obtained by

adding even multiples of π to it or subtracting even multiples of π from it. Thus XOP represents the angles

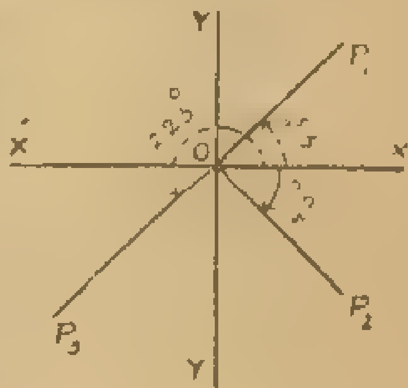
α, $2\pi+\alpha$, $4\pi+\alpha$, $6\pi+\alpha$, $8\pi+\alpha$,.........etc.
α, $-2\pi+\alpha$, $-4\pi+\alpha$, $-6\pi+\alpha$, $-8\pi+\alpha$,...etc.

All these angles are included in expression $2n\pi+\alpha$, where n may be any positive or negative integer or zero.

Example 1. *If the revolving line OP coincides with OX, then all the angles so formed are expressed by* $2n\pi$.

Example 2. *What are the expressions to denote angles traced by OP when OP coincides with OY, OX' and OY'.*

$$\left[Ans: \; 2n\pi+\frac{\pi}{2};\; 2n\pi+\pi;\; 2n\pi+\frac{3\pi}{2}\right]$$



Example 3. *Find a general expression to denote all angles* $\angle XOP$ *when (i) the revolving line is in the position of* OP_1. $\left[Ans.\; 2n\pi+\frac{\pi}{4}\right]$

(ii) *The revolving line is in the position of* OP_2. $\left[Ans.\; 2n\pi+\frac{\pi}{4}\right]$

(iii) *The revolving line is in the position of* OP_3. $\left[Ans.\; 2n\pi+\frac{5\pi}{4} \text{ or } (2n+1)\pi+\frac{\pi}{4}\right]$

12. We revert to our question: 'what are the angles whose sign is $\frac{1}{2}$?' We may ask 'what are the angles whose sine is k?'; Similarly we may ask what are the angles whose cosine is m or whose tangent is t.

We are thus required to solve the equations
(*i*) $\sin\theta=k$
(*ii*) $\cos\theta=m$
(*iii*) $\tan\theta=t$.

§ 31. To find the most general expression for all angles having a given sine or a cosecant. (*P. U. 1942, 47*)

Let α be the least positive or negative angle having the given sine and θ any other angle having the same sine.

We have to find the most general value of θ which satisfies the equation

$$\sin\theta=\sin\alpha.$$

i.e., $\sin\theta-\sin\alpha=0$

$\therefore\ 2\cos\dfrac{\theta+\alpha}{2}.\sin\dfrac{\theta-\alpha}{2}=0$

$\therefore$ Either $\sin\dfrac{\theta-\alpha}{2}=0$ or $\cos\dfrac{\theta+\alpha}{2}=0$

where $\sin\dfrac{\theta-\alpha}{2}=0$, then $\dfrac{\theta-\alpha}{2}=0,\ \pi,\ 2\pi,\ 3\pi,\ldots\ldots$

$=$ any multiple of π

$\therefore\quad \dfrac{\theta-\alpha}{2}=p\pi$ or $\theta=2p\pi+\alpha\ldots$ ...(1)

when $\cos\dfrac{\theta+\alpha}{2}=0$, then $\dfrac{\theta+\alpha}{2}=\dfrac{\pi}{2},\ \dfrac{3\pi}{2},\ \dfrac{5\pi}{2},\ldots\ldots$

$=$ any odd multiple of $\dfrac{\pi}{2}$

$\therefore\quad \dfrac{\theta+\alpha}{2}=(2q+1)\dfrac{\pi}{2}$ or $\theta=(2q+1)(\pi-\alpha)$ ...(2)

From (1) and (2)

$$\theta=n\pi+(-1)^n\alpha.$$

EXAMPLES

Find the most general values of θ satisfying the equations:—

1. $\sin\theta=\dfrac{1}{\sqrt{2}}$.

[Hint : $\therefore\ \sin\theta=\sin\dfrac{\pi}{4}$ $\therefore$ the least value of $\theta=\dfrac{\pi}{4}$

$\therefore$ general value of $\theta=n\pi+(-1)^n.\ \dfrac{\pi}{4}$]

2. $\operatorname{cosec}\theta=\dfrac{2}{\sqrt{3}}$

[Hint : The equation reduces to $\dfrac{1}{\sin\theta}=\dfrac{2}{\sqrt{3}}$

or $\sin\theta=\dfrac{\sqrt{3}}{2}$ etc.]

3. Find the most general value of θ satisfying the equations.

(i) $\sin\theta=\frac{1}{2}$ (P.U.)

(ii) $\sin\theta=-\frac{1}{2}$.

[Hint : $\sin\theta=\frac{1}{2}=\sin 30°=\sin\frac{\pi}{6}$.

$\therefore \quad \theta=n\pi+(-1]^n\frac{\pi}{6}$.]

Note. It is to be noted that the angle is always expressed in radians.

9·4. To find the most general expression for all angles having a given cosine or a secant. (*P.U. 1944*)

Let α be the least positive angle having the given cosine and θ any other angle having the same cosine. We have to find the most general value of θ which satisfies the equation $\cos\theta=\cos\alpha$.

i.e., $\cos\theta-\cos\alpha=0$

or $2\sin\frac{\theta+\alpha}{2}\sin\frac{\theta-\alpha}{2}=0$

$\therefore$ Either $\sin\frac{\theta+\alpha}{2}=0$, or $\sin\frac{\theta-\alpha}{2}=0$.

when $\sin\frac{\theta-\alpha}{2}=0$, then $\frac{\theta-\alpha}{2}=0, \pi, 2\pi : 3\pi......$

$=$ano multiple of π

$\therefore \quad \frac{\theta-\alpha}{2}=p\pi \qquad \therefore \quad \theta=2p\pi+\alpha.........(1)$

when $\sin\frac{\theta+\alpha}{2}=0$, then $\frac{\theta+\alpha}{2}=0, \pi, 2\pi, 3\pi, 4\pi......$

$=$any multiple of π

$\therefore \quad \frac{\theta+\alpha}{2}=q\pi \qquad \therefore \quad \theta=2q\pi-\alpha.........(2)$

From (1) and (2) $\qquad \theta=2n\pi\pm\alpha$.

Note : To find a general expression for all angles having the same secant. we change secants on both sides into cosines.

i.e., If $\sec\theta=\sec\alpha$, we write $\frac{1}{\cos\theta}=\frac{1}{\cos\alpha}$ or $\cos\theta=\cos\alpha$.

EXAMPLES

1. What is the most general value of θ satisfying the equation. (1) $\cos\theta=-\frac{1}{2}$. (2) $\sec\theta=+\sqrt{2}$.

[**Hint :** (*i*) $\cos\theta=-\frac{1}{2}=\cos 120°=\cos\frac{2\pi}{3}$

$$\therefore\quad \theta=2n\pi\pm\frac{2\pi}{3}.$$

(*ii*) $\sec\theta=\sqrt{2}\quad \therefore\ \cos\theta=\frac{1}{\sqrt{2}}=\cos 45°=\cos\frac{\pi}{4}$

$$\therefore\quad \theta=2n\pi\pm\frac{\pi}{4}\ ]$$

2. Solve the simultaneous equation

$\cos(2x+3y)=\frac{1}{2}$ and $\cos(3x+2y)=\frac{\sqrt{3}}{2}$. (P. U. 1944)

[**Hint :** $\cos(2x+3y)=\frac{1}{2}=\cos 60°=\cos\frac{\pi}{3}$.

$$\therefore\quad 2x+3y=2n\pi\pm\frac{\pi}{3}\quad (1)$$

Similarly $3x+2y=2m\pi\pm\frac{\pi}{6}$ (2), where m, n are any integers 0, +ve or −ve :

From (1) and (2), find the value of x and y.]

9·5 To find the most general expression for all angles having a given tangent or a cotangent.

(*P.U. 1943-48*)

Let α be the least positive angle having the same tangent and θ any other angle having the same tangent. We have to find the most general value of θ, which satisfies the equation $\tan\theta=\tan\alpha$.

$$\therefore\quad \frac{\sin\theta}{\cos\theta}=\frac{\sin\alpha}{\cos\alpha}.$$

i.e., $\sin\theta.\cos\alpha-\cos\theta.\sin\alpha=0$

or $\quad\sin(\theta-\alpha)=0.$

$\therefore\quad \theta-\alpha=0,\ \pi,\ 2\pi,\ 3\pi,\ldots\ldots$

$=$ any multiple of $\pi=n\pi$.

$\therefore\quad \theta=n\pi+\alpha.$

Note : If $\cot\theta = \cot\alpha$, we write then

as $\frac{1}{\tan\theta} = \frac{1}{\tan\alpha}$ or $\tan\theta = \tan\alpha$.

$\therefore \quad \theta = n\pi + \alpha$.

EXAMPLES

1. What is the most general value of θ satisfying the equation (*i*) $\tan\theta = 1$

(*ii*) $\cot\theta = -\sqrt{3}$?

(*i*) $\tan\theta = 1 = \tan 45° = \tan\frac{\pi}{4} \qquad \left(\therefore \quad \alpha = \frac{\pi}{4}\right)$

$\therefore \quad \theta = n\pi + \frac{\pi}{4}$.

(*ii*) $\cot\theta = -\sqrt{3} \quad \therefore \tan\theta = -\frac{1}{\sqrt{3}} = \tan(-30°)$

$= \tan\left(-\frac{\pi}{6}\right)$

$\therefore \quad \theta = n\pi - \frac{\pi}{6}$.

2. Find the most general values of θ satisfying

(*i*) $\sin^2\theta = \frac{1}{4}$ [$\therefore \sin^2\theta = \frac{1}{4} = \sin^2\frac{\pi}{6}$

$\therefore \quad \theta = n\pi + (-1)^n\frac{\pi}{6}$]

(*P. U. 1943*)

(*ii*) $3\tan^2\theta = 1$

[$\therefore \quad \tan^2\theta = \pm\frac{1}{\sqrt{3}} = \tan\left(\pm\frac{\pi}{6}\right)$

$\therefore \quad \theta = n\pi \pm \frac{\pi}{6}$.

(*iii*) $2\cot^2\theta = \text{cosec}^2\theta$ (*P. U. 1942-S*)

[$\therefore 2\cot^2\theta = 1 + \cot^2\theta$

or $\quad \cot^2\theta = 1$ or $\cot\theta = \pm 1$

or $\quad \tan\theta = \pm 1 = \tan\left(\pm\frac{\pi}{4}\right)$

$\therefore \quad \theta = n\pi \pm \frac{\pi}{4}$.]

9·6 We shall now discuss four different types of Trigonometric Equation.

(*i*) Equations expressible in terms of one t-ratio of the unknown angle. (*P.U. 1950*)

(*ii*) Equations involving two multiple angles. (*P.U. 1942*)

(*iii*) Equations involving more than two multiple angles. (*P. U. 1935, 37*)

(*iv*) The classic equations $a \cos \theta + b \sin \theta = c$ and equations reducible to this form. (*P. U. 1941*)

There are also some miscellaneous equations, which though not explicitly of the Four Types given above, can, however, be brought to one of these forms :

We take up equations of First.

I Type. *Solve the equations* $8 \sin^2\theta - 2 \cos \theta = 5$. (*P. U. 1950*)

The equation may be written as

$8(1-\cos^2\theta) - 2 \cos \theta - 5 = 0$ which is now an equation expressed in one *t*-ratio.)

or $8 \cos^2\theta + 2 \cos \theta - 3 = 0$

$$\therefore \quad \cos \theta = \frac{-2 \pm \sqrt{4+96}}{16}$$

$$= \frac{-2 \pm 10}{16} \text{ or } +\frac{1}{2} \text{ or } -\frac{3}{4}.$$

Both the values are permissible as both are less than unity.

(*a*) $\therefore \cos \theta = \frac{1}{2} = \cos \frac{\pi}{3} \quad \therefore \quad \theta = 2n\pi \pm \frac{\pi}{3}$.

(*b*) $\cos \theta = -\frac{3}{4} = \cos (138° \; 25')$

$$= \cos\left(\frac{1661}{2160}\pi\right) \quad \therefore \theta = 2n\pi \pm \frac{1661}{2160}\pi.$$

or $\theta = \cos^{-1}(-\frac{3}{4}) \quad \therefore \theta$ in general value $= 2n\pi \pm \cos^{-1}(-\frac{3}{4})$.

Note : In the solution of a trigonometric equation the general value should always be given.

II Type. *Solve the equation*

$\sin n\theta = \cos m\theta$. (*P. U. 1942*)

Express cosine in terms of sine or sine in terms of cosine ; in this way the equation shall be expressed in one *t*-ratio.

1st Method : $\sin n\theta = \sin\left(\frac{\pi}{2} - m\theta\right)$

$$\therefore \quad n\theta = p\pi + (-1)^p\left(\frac{\pi}{2} - m\theta\right)$$

[we have used p' and not 'n' to avoid confusion with the given n of the equation.]

$\therefore$ By transposition

$$[n+(-1)^p m]\theta = p\pi + (-1)^p \cdot \frac{\pi}{2}$$

$$\therefore \quad \theta = \frac{p\pi + (-1)^p \frac{\pi}{2}}{n+(-1)^p m}$$

2nd Method : we express both sides in terms of cosines

$$\therefore \quad \cos\left(\frac{\pi}{2} - n\theta\right) = \cos m\theta$$

$$\therefore \quad \frac{\pi}{2} - n\theta = 2p\pi \pm m\theta$$

By transposition

$$(n \pm m)\theta = \frac{\pi}{2} - 2p\pi$$

$$\therefore \quad \theta = \frac{\frac{\pi}{2} - 2p\pi}{n \pm m}$$

III Type. *Solve the equation*

$\cos\theta - \cos 2\theta = \sin 3\theta$ *(P. U. 1935)*

Combing L.H.S. into factors.

$$\therefore \quad 2\sin\frac{3\theta}{2} \cdot \sin\frac{\theta}{2} = 2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2}$$

$$\text{or} \quad 2\sin\left(\frac{3\theta}{2}\right)\left[\sin\frac{\theta}{2} - \cos\frac{3\theta}{2}\right] = 0$$

$$\therefore \quad \sin\frac{3\theta}{2} = 0 \; (1) \text{ or } \sin\frac{\theta}{2} - \cos\frac{3\theta}{2} = 0 \qquad (2)$$

From (1) $\sin\frac{3\theta}{2} = \sin(0) \quad \therefore \quad \frac{3\theta}{2} = n\pi \quad \therefore \theta = \frac{2n\pi}{3}$

From (2) $\sin\frac{\theta}{2}=\cos\frac{3\theta}{2}=\sin\left(\frac{\pi}{2}-\frac{3\theta}{2}\right)$

(which is of the form of II Type)

$$\therefore \quad \frac{\theta}{2}=n\pi+(-1)^n\left[\frac{\pi}{2}-\frac{3\theta}{2}\right]$$

By transposition

$$\theta+(-1)^n.3\theta=2n\pi+(-1)^n\pi$$

$$\therefore \quad \theta=\frac{2n\pi+(-1)^n\pi}{1+(-1)^n.3}$$

IV Type. *The Classic Equation.*

To solve the equation $a\cos\theta+b\sin\theta=c$ ...(1)

1st Method. Put $r\cos\alpha=a$ and $r\sin\alpha=b$

Squaring and adding $r=\pm\sqrt{a^2+b^2}$

Dividing we have $\tan\alpha=\frac{b}{a}$ or $\alpha=\tan^{-1}\left(\frac{b}{a}\right)$

Now r is usually taken as positive and this gives one value of α consistent with (1). The value of α can be found from the tables. The equation becomes

$$r\cos\alpha\cos\theta+r\sin\alpha\sin\theta=c$$

or $\quad r\cos(\theta-\alpha)=c$

i.e., $\quad \cos(\theta-\alpha)=\frac{c}{r}=\frac{c}{\sqrt{a^2+b^2}}=\cos(\beta)$ say

$\therefore \quad \theta-\alpha=2n\pi\pm\beta$

$\therefore \quad \theta=2n\pi+\alpha\pm\beta$

Note 1. The solution of this equation is possible only if $c<\sqrt{a^2+b^2}$, otherwise the value of β connot be found out.

Note 2. The equation can also be evolved by taking

$$r\cos\alpha=b \text{ and } r\sin\alpha=a$$

2nd Method. By putting $\tan\frac{\theta}{2}=t$

$$\sin\theta=\frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{2t}{1+t^2}$$

$$\text{and }\cos\theta=\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{1-t^2}{1+t^2}$$

The equation becomes $a\dfrac{1-t^2}{1+t^2}+b\dfrac{2t}{1+t^2}=c$

or $(c+a)t^2-2bt+(c-a)$

$\therefore \quad t=\dfrac{b\pm\sqrt{a^2+b^2-c^2}}{c+a}$

The solution is possible only if $a^2+b^2>c^2$. If it is so then there are two values of the angle say $\dfrac{\alpha}{2}$ and $\dfrac{\beta}{2}$

$\therefore \quad \tan\dfrac{\theta}{2}=\tan\dfrac{\alpha}{2}$ or $\tan\dfrac{\beta}{2}$

$\therefore \quad \dfrac{\theta}{2}=n\pi+\dfrac{\alpha}{2}$ or $n\pi+\dfrac{\beta}{2}$

$\therefore \quad \theta=2n\pi+\alpha$ or $2n\pi+\beta$.

Example. *Solve*.—$\sqrt{3}\sin\theta+\cos\theta=\sqrt{2}$

Put $r\cos\alpha=1$ and $r\sin\alpha=\sqrt{3}$

$\therefore \quad r=2$ and $\tan\alpha=\sqrt{3} \quad \therefore \quad \alpha=\dfrac{\pi}{3}$

The equation becomes $r\left(\cos\theta\cos\dfrac{\pi}{3}+\sin\theta\sin\dfrac{\pi}{3}\right)=\sqrt{2}$

or $\cos\left(\theta-\dfrac{\pi}{3}\right)=\dfrac{\sqrt{2}}{2}=\dfrac{1}{\sqrt{2}}=\cos\dfrac{\pi}{4}$

$\therefore \quad \theta-\dfrac{\pi}{3}=2\pi\pm\dfrac{\pi}{4}$

or $\theta=2n\pi+\dfrac{\pi}{3}\pm\dfrac{\pi}{4}$.

MISCELLANEOUS EQUATIONS

9.7. To solve $a\sin^2\theta+2b\sin\theta\cos\theta+\cos^2\theta=d$.

It is an equation *homogeneous* in $\sin\theta$ and $\cos\theta$

To solve this divide throughout by $\cos^2\theta$ and put $\tan\theta=t$

$\therefore \quad at^2+2bt+c=d(1+t^2)$

or $(a-d)t^2+2bt+c-d=0$

Solving for t, we get $\tan\theta=t=\dfrac{-b\pm\sqrt{b^2-(a-d)(c-d)}}{(a-d)}$

If α and β are two angles which satisfy the equation,

$$\theta=n\pi+\alpha \text{ or } n\pi+\beta$$

Example. *Solve :—*

$$\operatorname{cosec}^2\theta-\tfrac{2}{3}\sqrt{3}\operatorname{cosec}\theta\sec\theta-\sec^2\theta=0$$

This can be written as

$$\frac{1}{\sin^2\theta}-\frac{2\sqrt{3}}{3\sin\theta\cos\theta}-\frac{1}{\cos^2\theta}=0$$

$$\cos^2\theta-\frac{2\sqrt{3}}{3}\sin\theta\cos\theta-\sin^2\theta=0$$

Dividing by $\cos^2\theta$ we get

$$1-\frac{2\sqrt{3}}{3}\tan\theta-\tan^2\theta=0$$

$$\text{or } \tan^2\theta+\frac{2\sqrt{3}}{3}\tan\theta-1=0$$

$$\tan\theta=\frac{\dfrac{-2\sqrt{3}}{3}\pm\sqrt{\dfrac{12}{9}+4}}{2}=-\frac{\sqrt{3}}{3}\pm\frac{2\sqrt{3}}{3}=\frac{1}{\sqrt{3}}$$

or $\quad -\sqrt{3}$

$$=\tan\frac{\pi}{6} \text{ or } \tan\left(-\frac{\pi}{3}\right)$$

$$\therefore\quad \theta=n\pi+\frac{\pi}{6} \text{ or } n\pi-\frac{\pi}{3}$$

We give below some more solved examples to show that there are more than one method for the solution of the same equation and also to show that some bigger equations can be reduced to simpler ones.

Example 1. *Solve the equation* $\sin 2x=\sin x$

1st Method. $\quad \sin 2x=\sin x$

$$\therefore\quad \sin 2x-\sin x=0$$

$$\therefore\quad 2\cos\frac{3x}{2}\sin\frac{x}{2}=0$$

$$\therefore\quad \text{Either } \cos\frac{3x}{2}=0=\cos\frac{\pi}{2}$$

$$\text{whence } \frac{3x}{2}=2n\pi+\frac{\pi}{2}$$

$$\text{or}\quad \sin\frac{x}{2}=0=\sin 0$$

$$\text{whence } \frac{x}{2}=n\pi.$$

2nd Method. $\sin 2x = \sin x$

$\therefore\ 2\sin x\cos x - \sin x = 0$

$\therefore\ \sin x\,(2\cos x - 1) = 0$

$\therefore$ Either $\sin x = 0 = \sin 0$, whence $x = n\pi$

or $\cos x = \frac{1}{2} = \cos\frac{\pi}{3}$, whence $x = 2n\pi \pm \frac{\pi}{3}$

$\therefore\ x = n\pi$ or $2n\pi \pm \frac{\pi}{3}$.

3rd Method. $\sin 2x = \sin x$

$\therefore\ 2x = n\pi + (-1)^n x$

$\therefore\ x\,\{2 - (-1)^n\} = n\pi$

$$\therefore\ x = \frac{n\pi}{2-(-1)^n} = \frac{n\pi\,\{2+(-1)^n\}}{4-(-1)^{2n}} = \frac{n\pi}{3}\{2+(-1)^n\}$$

Note. The discrepancies in the results are due to the different steps of procedure.

Example 2. *Solve the equation*

$$\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$$

Since $\sin x + \sin 2x + \sin 3x = \cos x + \cos 2x + \cos 3x$

$\therefore\ \sin x + \sin 3x + \sin 2x = \cos x + \cos 3x + \cos 2x$

$\therefore\ 2\sin 2x\cos x + \sin 2x = 2\cos 2x\cos x + \cos 2x$

$\therefore\ \sin 2x(2\cos x + 1) - \cos 2x(2\cos x + 1) = 0$

or $(2\cos x + 1)(\sin 2x - \cos 2x) = 0$

Either $\cos x = -\frac{1}{2} = \cos\frac{2\pi}{3}$

whence $x = 2n\pi \pm \frac{2\pi}{3}$;

or $\sin 2x - \cos 2x = 0$ whence $\tan 2x = 1 = \tan\frac{\pi}{4}$

$\therefore\ 2x = n\pi + \frac{\pi}{4}$

$\therefore\ x = 2n\pi \pm \frac{2\pi}{3}$ or $\frac{n\pi}{2} + \frac{\pi}{8}$

Example 3. *Solve the equation*

$$\cos^3\theta\sin 3\theta + \sin^3\theta\cos 3\theta = \frac{3\sqrt{3}}{8}$$

1st Method. $\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\therefore \quad \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\therefore \quad \cos^3\theta = \frac{\cos 3\theta + 3\cos\theta}{4}$$

$\therefore$ The equation becomes

$$\frac{\cos 3\theta + 3\cos\theta}{4}\sin 3\theta + \frac{3\sin\theta - \sin 3\theta}{4}.\cos 3\theta = \frac{3\sqrt{3}}{8}$$

$$\therefore \quad \frac{3}{4}(\sin 3\theta\cos\theta + \sin\theta\cos 3\theta) = \frac{3\sqrt{3}}{8}$$

$$\text{or} \quad \sin(3\theta + \theta) = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$$

$$\therefore \quad 4\theta = n\pi + (-1)^n\frac{\pi}{3} \quad \text{or } \theta = \frac{1}{4}\left\{n\pi + (-1)^n\frac{\pi}{3}\right\}$$

2nd Method. $\cos^3\theta\sin 3\theta + \sin^3\theta\cos 3\theta = \dfrac{3\sqrt{3}}{8}$

$$\therefore \cos^3\theta(3\sin\theta - 4\sin^3\theta) + \sin^3\theta(4\cos^3\theta - 3\cos\theta) = \frac{3\sqrt{3}}{8}$$

$$\therefore \quad 3\sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta) = \frac{3\sqrt{3}}{8}$$

$$\text{or } 4\sin\theta\cos\theta\cos 2\theta = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \sin 4\theta = \frac{\sqrt{3}}{2} = \sin\frac{\pi}{3}$$

$$\therefore \quad 4\theta = n\pi + (-1)^n\frac{\pi}{0}$$

$$\therefore \quad \theta = \frac{n\pi}{4} + (-1)^n\frac{\pi}{12}$$

9·8. To find the most general value of an angle satisfying two equations :

Example. 1. *What is the most general value of θ which satisfies both the equations.*

(*i*) $\sin\theta = -\frac{1}{2}$ *and* (*ii*) $\tan\theta = \dfrac{1}{\sqrt{3}}$ (*P.U. 1942*)

Sol. We shall consider the value of θ lying between 0 and 360°, which shall satisfy both these equations. In order to make it general we shall add $2n\pi$ to it.

$\sin\theta = \frac{1}{2} = -\sin(30°) = \sin(180° + 30)$ or $= \sin(360° - 30°)$

$\therefore \quad \theta = 210°$ or $330°)$ ... (1)

Again $\tan\theta = \frac{1}{\sqrt{3}} = \tan 30°$ or $= \tan(180° + 30°) = \tan 210°$.

$\therefore$ The value of θ (from $\tan\theta$) $= 30°$ or $210°$

$\therefore$ The value of θ satisfying both the equations

$$\text{is } 210° \text{ or } \frac{7\pi}{6}$$

$\therefore$ The general value of $\theta = 2n\pi + \frac{7\pi}{6}$.

Example 2. *Find the most general value of θ which satisfies the equations* $\sin\theta = \frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$.

Considering the angles between 0 and 360°, the only values of θ which satisfy $\sin\theta = \frac{1}{2}$ are 30° and 150°.

Similarly the only values of θ which satisfy $\cos\theta = \frac{\sqrt{3}}{2}$ are 150° and 210°. The prime value of θ which satisfies both is 150° or $\frac{5\pi}{6}$.

$\therefore \theta = 2n\pi + \frac{5\pi}{6}$.

9·9. Solution of Simultaneous Equations.

Example 1. *Solve* $\cos(\theta - \phi) = \frac{\sqrt{3}}{2}$, $\tan(2\theta + \phi) = 1$.

We have $\cos(\theta - \phi) = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$ ... (1)

$\tan(2\theta + \phi) = 1 = \tan\frac{\pi}{4}$ ... (2)

$\therefore$ From (1) $\theta - \phi = 2n\pi \pm \frac{\pi}{6}$ ... (3)

$\therefore$ From (2) $2\theta + \phi = m\pi + \frac{\pi}{4}$ ... (4)

Hence from (3) and (4) $\theta = \frac{1}{3}\left\{2n\pi \pm \frac{\pi}{6} + m\pi + \frac{\pi}{4}\right\}$

$$\phi = \tfrac{1}{2}\left\{m\pi + \frac{\pi}{4} - 2n\pi \pm \frac{\pi}{6}\right\}$$

EXERCISES IX (C)

Solve the following equations, giving the most general values of the variable.

Note. Questions (1 – 6) are of I type.

1. $2\cos^2 x + 3\sqrt{3}\sin x - 5 = 0.$ *(P. U. 1949)*

2. $2\sin^2 x + 3\cos x - 3 = 0.$ *(P. U. (1947-S)*

3. $3\tan\theta + \cot\theta = 5\operatorname{cosec}\theta.$ *(P. U. 1941)*

[**Hint** : Changing into sines and cosines, we get

$$\frac{3\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{5}{\sin\theta} \text{ or } 3\sin^2\theta + \cos^2\theta = 5\cos\theta].$$

4. $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta.$ *(D. U. 1950)*

5. $\tan^2\theta + \cot^2\theta = 2.$

[Write $\cot\theta = \frac{1}{\tan\theta}$, we get $\tan^2\theta + \frac{1}{\tan^2\theta} = 2$

or $\tan^4\theta - 2\tan^2\theta + 1 = 0$ or $(\tan^2\theta - 1)^2 = 0.$]

6. (*i*) $\tan^2\theta + \sec\theta - 1 = 0$ (*ii*) $\operatorname{cosec}\theta - \cot^2\theta - 1 = 0.$

Note. Questions 7 – 10 of II Type.

7. (*i*) $\sin 5\theta = \sin 3\theta.$

(*ii*) $\cos p\theta = \cos q\theta.$

(*iii*) $\cos 3\theta = \sin 2\theta.$

8. (*i*) $\cos a\theta + \cos b\theta = 0.$

(*ii*) $\sin m\theta + \cos n\theta = 0.$

[**Hint** : (*i*) $\cos a\theta = -\cos b\theta = \cos(\pi - b\theta).$

(*ii*) $\cos n\theta = -\sin m\theta = \cos\left(\frac{\pi}{2} + m\theta\right)$]

9. (*i*) $\cot A = \tan 2A.$ *(P. U. 1944-S)*

(*ii*) $\tan 5\theta = \cot 2\theta.$ *(P. U. 1943)*

[**Hint** : (*i*) $\tan 2A = \cot A = \tan(90 - A)$]

10. $\tan(\pi \cot\theta)=\cot(\pi\tan\theta)$. (*P. U. 1938*)

[**Hint**: $\tan(\pi\cot\theta)=\cot(\pi\tan\theta)=\tan\left(\frac{\pi}{2}-\pi\tan\theta\right)$

$$\therefore\quad \pi\cot\theta=n\pi+\frac{\pi}{2}-\pi\tan\theta$$

or $\qquad \cot\theta=n+\frac{1}{2}-\tan\theta$

or $\quad \tan\theta+\cot\theta=n+\frac{1}{2}$ etc.]

Note. Questions 11–16 are of III Type.

11. $\sin x+\cos 3x=\cos 5x$. (*P. U. 1947*)
12. $\sin 7\theta+\sin 3\theta=\sin 5\theta$. (*P. U. 1942*)
13. $\sin\theta+\sin 2\theta+\sin 3\theta=0$. (*P. U. 1945*)
14. $\sin x+\sin 3x=\sin 2x$.
15. $\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta=0$. (*P. U. 1937*)

[**Hint**: $(\cos\theta+\cos 7\theta)+(\cos 3\theta+\cos 5\theta)=0$
or $2\cos 4\theta.\cos 3\theta+2\cos 4\theta.\cos\theta=0$
or $2\cos 4\theta\{\cos 3\theta+\cos\theta\}=0$]

16. $\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta=0$.

Note. Questions 17–21 are of IV type.

17. $\cos\theta+\sin\theta=\sqrt{2}$. (*P. U. 1941*)

[Dividing both sides by $\sqrt{1^2+1^2}$ or $\sqrt{2}$ we get

$$\frac{1}{\sqrt{2}}.\cos\theta+\frac{1}{\sqrt{2}}\sin\theta=1$$

$\therefore\ \cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}.\sin\theta=1$

$\therefore\ \cos\left(\theta-\frac{\pi}{4}\right)=\cos 0$

$\therefore\ \theta-\frac{\pi}{4}=2n\pi\pm 0$

$\therefore\ \theta=2n\pi+\frac{\pi}{4}$

or $\sin\frac{\pi}{4}.\cos\theta+\cos\frac{\pi}{4}\sin\theta=1$

$\therefore\ \sin\left(\theta+\frac{\pi}{4}\right)=\sin\frac{\pi}{2}$

$\therefore\ \theta+\frac{\pi}{4}=n\pi+(-1)^n.\frac{\pi}{2}$

$\therefore\ \theta=n\pi+(-1)^n.\frac{\pi}{2}-\frac{\pi}{4}$.

Note. Both the results, though different, can be shown to be identical.

If n is even (say $=2m$), the second result is

$$=2m\pi+(-1)^{2m}.\frac{\pi}{2}-\frac{\pi}{4}=2m\pi+\frac{\pi}{4}.$$

Again if $n=2m+1$ (odd), the same result

$$=(2m+1)\pi+(-1)^{2m+1}.\frac{\pi}{2}-\frac{\pi}{4}=2n.\pi+\frac{\pi}{4}.$$

Thus these results are the same as the first result. We observe that trigonometrical equations can be solved in different ways, but the results can always be proved to be identical.]

18. (*i*) $\sqrt{3}\sec\theta+\tan\theta=1.$ (*P. U. 1933*)
(Change into sines and cosines). (*P. U. 1940*)
(*ii*) $\operatorname{cosec}\theta=\sqrt{3}+\cot\theta$

19. $\sqrt{3}\sin\theta-\cos\theta=\sqrt{2}.$ (*D. U.*)

20. $\cos\theta+\sqrt{3}\sin\theta=2.$ (*C. U.*)

21. $\cos\theta+\sqrt{3}\sin\theta=\dfrac{\sqrt{5}+1}{2}.$

[Divide throughout by $\sqrt{1^2+(\sqrt{3})^2}$]

22. $\cos^3\theta-\cos\theta.\sin\theta-\sin^3\theta=1.$

23. Find the general value of θ, which may satisfy the equations.

(*i*) $\sin\theta=\frac{1}{2}$ and $\cos\theta=\frac{\sqrt{3}}{2}.$

(*ii*) $\cos\theta=-\frac{\sqrt{3}}{2}$ and $\cot\theta=-\sqrt{3}.$

(*iii*) $\sec\theta=\sqrt{2}$ and $\tan\theta=-1.$

24. Prove that all angles, which are both equi-sinal and equi-cosinal with α are included in the formula $2n\pi+\alpha.$

[**Hint** : $\sin\theta=\sin\alpha$	$\cos\theta=\cos\alpha$
$=\sin(180^\circ-\alpha)$	$=\cos(360^\circ-\alpha)$
$\therefore\quad\theta=\alpha$ or $180^\circ-\alpha$	$\therefore\quad\theta=\alpha$ or $360^\circ-\alpha$

$\therefore$ Common value of $\theta=\alpha$

Hence general value of $\theta=2n\pi+\alpha.$]

25. Solve the simultaneous equations.

(*i*) $\sin(\theta-\phi)=\frac{1}{2}$ and $\cos(\theta+\phi)=\frac{1}{2}.$

(*ii*) $\cos\theta.\cos\phi=\frac{1}{4}$, and $\sin\theta.\sin\phi=\frac{\sqrt{5}}{4}.$

(*iii*) $\tan(x-y)=1$ and $\cos(x+y)=\frac{\sqrt{3}}{2}.$

MISCELLANEOUS EXERCISES ON CHAPTER IX

1. Prove that in any triangle $\sin A \sin (B-C)$
$$+\sin B.\sin (C-A)+\sin C.\sin (A-B)=0.$$

2. Prove that in any triangle
$$\cos 4A+\cos 4B+\cos 4C=-1+4 \cos 2A.\cos 2B.\cos 2C.$$

If $A+B+C=180°$, *prove that*

3. $\cot A+\cot B+\cot C-\cot A.\cot B.\cot C.$
$$=\operatorname{cosec} A.\operatorname{cosec} B.\operatorname{cosec} C. \qquad (D\ U.)$$

[**Hint :** $\cos (A+B+C)=-1$. Expand and divide by **sin A.sin B.sin C.**

4. $$\frac{\cot A+\cot B}{\tan A+\tan B}+\frac{\cot B+\cot C}{\tan B+\tan C}+\frac{\cot C+\cot A}{\tan C+\tan A}=1.$$
(*P.U. 1931*)

5. $$\sin 3A+\sin 3B+\sin 3C=-4 \cos \frac{3A}{2}.\cos \frac{3B}{2}.\cos \frac{3C}{2}.$$

6. If $\alpha+\beta=\gamma$, show that
$$\cos^2\alpha-\cos^2\beta+\cos^2\gamma=1+2 \cos \alpha \cos \beta \cos \gamma. \qquad (C.U.)$$

[**Hint :** $\cos (\alpha+\beta)=\cos \gamma \therefore \cos \alpha.\cos \beta-\sin \alpha.\sin \beta=\cos \gamma$
or $\cos \alpha \cos \beta-\cos \gamma=\sin \alpha.\sin \beta$

Squaring and simplifying we get the required relation].

7. If $x+y+z=xyz$, prove that
$$\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}.$$

[**Hint :** Proceed as in Solved Ex. 7 before Exercise IX (A).]

8. Prove that
$$\tan 24°+\tan 75°+\tan 81°=\tan 24°.\tan 75°.\tan 81°$$
(*P. U. 1938*)

[**Hint :** $\tan (24°+75°)=\tan (180-81°)=-\tan 81°$.

or $\dfrac{\tan 24°+\tan 75°}{1-\tan 24° \tan 75°}=-\tan 81°.$]

Cross-multiply.

9. Prove that

(*i*) $\tan 17°.\tan 40°+\tan 40°.\tan 33°+\tan 33°.\tan 17°=1.$

(*ii*) $\tan \dfrac{\pi}{3}.\tan \dfrac{\pi}{8}+\tan \dfrac{\pi}{8}.\tan \dfrac{\pi}{24}+\tan \dfrac{\pi}{24} \tan \dfrac{\pi}{3}=1.$

10. If $\cot\dfrac{A}{2}$, $\cot\dfrac{B}{2}$, $\cot\dfrac{C}{2}$ are in A. P.

Show that $\cot\dfrac{A}{2}.\cot\dfrac{C}{2}=3.$ *(P. U. 1935)*

[**Sol.** $$\cot\frac{A}{2}+\cot\frac{C}{2}=2\cot\frac{B}{2}=2\cot\left(\frac{\pi}{2}-\overline{\frac{A}{2}+\frac{C}{2}}\right)$$
$$=2\tan\left(\frac{A}{2}+\frac{C}{2}\right)$$
$$=\frac{2\left(\cot\frac{A}{2}+\cot\frac{C}{2}\right)}{\cot\frac{A}{2}.\cot\frac{C}{2}-1}$$

Cross-multiply.]

11. Show that $\sin(B-C)+\sin(C-A)+\sin(A-B)$
$$=-4\sin\frac{B-C}{2}.\sin\frac{C-A}{2}.\sin\frac{A-B}{2}.$$

12. If $x+y+z=xyz$, show that
$$x(1-y^2)(1-z^2)+y(1-z^2)(1-x^2)+z(1-x^2)(1-y^2)=4xyz.$$

Sol. Divide the identity in Sol. Ex. 7 Exercise IX (A) by 2 and multiply by $(1-x^2)(1-y^2)(1-z^2)$]

13. Solve the following equations.

(*i*) $\operatorname{cosec}^3 2\theta=4\operatorname{cosec} 2\theta$ *(P. U. 1941)*

(*ii*) $\cos\theta+\tan\theta=\sec\theta.$ *(C. U.)*

(*iii*) $\cos 3\theta+8\cos^3\theta=0.$ *(P. U. 1937)*

14. Find the value of θ in the following equations and write down those values which are $<2\pi$ and >0.

(*i*) $\cos 2x-\sin x-1=0.$ *(D. U. 1936)*

(*ii*) $\cos\theta+\cos 3\theta=\cos 2\theta.$ *(D. U. 1940)*

[**Hint :** (*ii*) The general solution as
$$\theta=2n\pi\pm\frac{\pi}{3}\text{ or }(2n+1)\frac{\pi}{4}.$$

Put $n=0, 1, 2, 3$......etc., till θ is $\geqslant 2\pi$

$\therefore\quad \theta=\pm\dfrac{\pi}{3},\ \dfrac{5\pi}{4}$ or $\dfrac{\pi}{4},\ \dfrac{3\pi}{4},\ \dfrac{5\pi}{4},\ \dfrac{7\pi}{4}.$]

15. Write down all positive values of θ less than $180°$, which satisfy the equations.

(*i*) $\sin 7\theta+\sin 3\theta=\sin 5\theta$. (*D.U. 1942*)

(*ii*) $\cos x+\cos 2x+\cos 3x=0$. (*P U. 1946*)

16. Solve the following equations :

(*i*) $\cot x-\tan x=2$.

(*ii*) $\sin 2x=3\tan x\cos 2x$ (*P.U. 1932*)

[Divide by $\cos 2x$ and change the result as a quadratic in $\tan x$.]

17. If $\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$, prove that

$$\cos\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$$ (*P.U. 1939*)

[**Sol.** $\tan(\pi\cos\theta)=\tan\left(\frac{\pi}{2}-\pi\sin\theta\right)$

$$\therefore\quad \pi\cos\theta=\frac{\pi}{2}-\pi\sin\theta$$

or $$\cos\theta+\sin\theta=\tfrac{1}{2}$$

$$\therefore\quad \frac{1}{\sqrt{2}}.\cos\theta+\frac{1}{\sqrt{2}}.\sin\theta=\frac{1}{2\sqrt{2}}$$

or $$\cos\theta.\cos\frac{\pi}{4}+\sin\theta.\sin\frac{\pi}{4}=\frac{1}{2\sqrt{2}}$$

$$\therefore\quad \cos\left(\theta-\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}$$]

18. Solve the equation $\tan\left(\frac{\pi}{4}+\theta\right)=3\tan\left(\frac{\pi}{4}-\theta\right)$ (*P.U. 1933*)

[**Hint :** $$\frac{1+\tan\theta}{1-\tan\theta}=3\,\frac{1-\tan\theta}{1+\tan\theta}$$

or $\left(\frac{1+\tan\theta}{1-\tan\theta}\right)=3$ or $\tan^2\left(\frac{\pi}{4}+\theta\right)=3$ etc.]

Solve the following equations (19 – 24)

19. $\tan^2\theta+(\sqrt{3}+1)\tan\theta+\sqrt{3}=0$.

20. $\cos\theta=2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$. (*P.U.*)

(*P.U.*)

21. $\tan\left(\frac{\pi}{4}-\theta\right)+\cot\left(\frac{\pi}{4}-\theta\right)=4.$

22. $\tan\theta+\tan 2\theta+\sqrt{3}\tan\theta\tan 2\theta=\sqrt{3}.$

[**Hint** : Change into sines and cosines.]

23. $\sin\theta+\cos\theta=\sqrt{2}\sin 2\theta.$

24. $3\cos^2\theta-2\sqrt{3}\sin\theta\cos\theta-3\sin^2\theta=0.$

[Divide throughout by $\cos^2\theta$ and we get a quadratic in $\tan\theta$.]

25. If $\cos p\theta+\cos q\theta=0$, prove that the different values of θ form two A P's, in which the common differences are

$$\frac{2\pi}{p+q} \text{ and } \frac{2\pi}{p-q} \text{ respectively.}$$

26 Solve the equation

$$4\sin(\theta+60^\circ).\sin(\theta-60^\circ)=\sin\theta$$

and write down all values of θ lying between 0° and 360°.

27. If $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$, show that

$$2\theta=\pm\sin^{-1}(\tfrac{3}{4}).$$

ANSWERS TO EXERCISES IN CHAPTER IX

Exercises IX (B)

1. $30^\circ, 150^\circ$. **2.** $0, 60^\circ, 180^\circ, 300^\circ$. **3.** 135°.
4. $0^\circ, 60^\circ, 120^\circ, 180^\circ$. **5.** $0^\circ, 45^\circ, 135^\circ, 180^\circ$. **6.** 9°.
7. $45^\circ, 135^\circ, 60^\circ, 300^\circ$. **8.** $120^\circ, 150^\circ$.
9. $31^\circ\ 19'$. **10.** $60^\circ\ 40'$.

Exercises IX (C)

1. $n\pi+(-1)^n\frac{\pi}{3}$. **2.** $2n\pi$ or $2n\pi\pm\frac{\pi}{3}$. **3.** $2n\pi\pm\frac{\pi}{3}$.

4. $2n\pi\pm\frac{\pi}{3}$. **5.** $n\pi\pm\frac{\pi}{4}$. **6.** (*i*) $2n\pi$ or $2n\pi\pm\frac{2\pi}{3}$.

(*ii*) $n\pi+(-1)^n\frac{\pi}{6}$ or $n\pi-(-1)^n\frac{\pi}{2}$.

7. (*i*) $\frac{n\pi}{5-(-1)^n.3}$. (*ii*) $\frac{2n\pi}{p\pm q}$. (*iii*) $\frac{n\pi+(-1)^n.\frac{\pi}{2}}{2+(-1)^n.3}$.

8. (i) $\dfrac{(2n\pm1)\pi}{a\pm b}$. (ii) $\dfrac{2k\pi\pm\dfrac{\pi}{2}}{n\pm m}$.

9. (i) $\dfrac{1}{3}\left(\dfrac{\pi}{2}-n\pi\right)$. (ii) $\dfrac{1}{7}\left(n\pi+\dfrac{\pi}{2}\right)$.

10. $\tan\theta=\frac{1}{4}[(2n+1)\pm\sqrt{4n^2+4n-15}]$.

11. $n\pi$ or $\dfrac{1}{4}\left[n\pi+(-1)^n.\dfrac{\pi}{6}\right]$ 12. $\dfrac{n\pi}{5}$ or $\dfrac{1}{2}\left[2n\pm\dfrac{\pi}{3}\right]$.

13. $\dfrac{n\pi}{2}$ or $2n\pi\pm\dfrac{2\pi}{3}$. 14. $\dfrac{n\pi}{2}$ or $2n\pi\pm\dfrac{\pi}{3}$.

15. $\theta=(2n+1)\dfrac{\pi}{8}$ or $(2n+1)\dfrac{\pi}{4}$ or $(2n+1)\dfrac{\pi}{2}$.

16. $\theta=2n\pi\pm\dfrac{\pi}{2}$ or $\dfrac{1}{2}\left(2n\pi\pm\dfrac{\pi}{2}\right)$ or $\dfrac{n\pi}{4}$.

18. (i) $2n\pi-\dfrac{\pi}{4}$. (ii) $2n\pi\pm\dfrac{\pi}{3}+\dfrac{\pi}{3}$.

19. $n\pi+(-1)^n\dfrac{\pi}{4}+\dfrac{\pi}{6}$. 20. $2n\pi\pm\dfrac{\pi}{3}$.

21. $2n\pi\pm\dfrac{\pi}{5}+\dfrac{\pi}{3}$. 22. $2n\pi$, $2n\pi-\dfrac{\pi}{2}$

23. (i) $2n\pi+\dfrac{\pi}{6}$. (ii) $2n\pi+\dfrac{5\pi}{6}$ (iii) $2n\pi-\dfrac{\pi}{4}$.

25. (i) $\theta=\left(m+\dfrac{n}{2}\right)\pi\pm\dfrac{\pi}{6}+(-1)^n.\dfrac{\pi}{12}$.

$$\phi=\left(m-\dfrac{n}{2}\right)\pi\pm\dfrac{\pi}{6}-(-1)^n\dfrac{\pi}{12}.$$

(ii) $\theta+\phi=2n\pi\pm\dfrac{3\pi}{5}$.

$$\theta-\phi=2m\pi\pm\dfrac{\pi}{5}.$$

(iii) $x-y=n\pi+\dfrac{\pi}{4}$.

$$x+y=2m\pi\pm\dfrac{\pi}{9}.$$

Miscellaneous Exercises on Chapter IX

13. (*i*) $\frac{n\pi}{2} \pm \frac{\pi}{12}$. (*ii*) $n\pi$ or $n\pi + (-1)^n \frac{\pi}{3}$.

(*iii*) $(2n+1)\frac{\pi}{2}$ or $n\pi \pm \frac{\pi}{3}$.

14. (*i*) $\theta = n\pi$ or $n\pi + (-)^n . \frac{7\pi}{6}$.

$\therefore$ Between 0 and 2π $\theta = 0, \pi, 2\pi, \frac{7\pi}{6}, \frac{11\pi}{6}$.

15. (*i*) 0, 36°, 72°, 108°, 144°, 30°, 150° :

(*ii*) $(2n+1)\frac{\pi}{4}$ or $2n\pi \pm \frac{3\pi}{4}$ $\left(\therefore \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}\right)$.

16. (*i*) $\frac{n\pi}{2} + \frac{\pi}{8}$. (*ii*) $n\pi$ or $n\pi \pm \frac{\pi}{6}$.

18. $n\pi \pm \frac{\pi}{3} - \frac{\pi}{4}$. **19.** $n\pi + \frac{3\pi}{4}$.

20. $n\pi$. **21.** $n\pi \pm \frac{\pi}{6}$. **22.** $\left(n + \frac{1}{3}, \frac{\pi}{3}\right)$.

23. $n\pi + \frac{\pi}{4}$. **24.** $\theta = n\pi - \frac{\pi}{3}$ or $n\pi + \frac{\pi}{6}$.

26. $\theta = 90°, 228° 35', 311° 25'$.

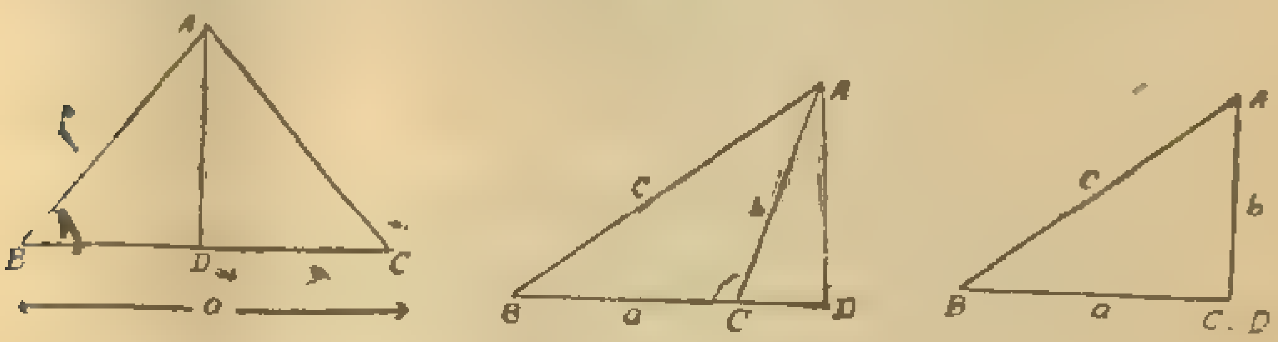
CHAPTER X

1. RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE

10 1. The Sine Formula. The sides of a triangle are proportional to the sines of the opposite angles, *i.e.*, in any $\triangle ABC$,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \qquad (P.U.\ 1926,\ 32,\ 35,\ 37,\ 40)$$

Let ABC be any triangle, one angle of which say $\angle C$ may be acute, obtuse or a right angle, and let $AD \perp BC$ or BC produced.



In all cases $\sin B=\dfrac{AD}{AB}$ $\therefore$ $AD=c \sin B$ ......(1)

when $\angle C$ is acute $\dfrac{AD}{AC}=\sin C$ $\therefore$ $AD=b \sin C$

when $\angle C$ is obtuse $\dfrac{AD}{AC}=\sin(180°-C)=\sin C$

$$\therefore \quad AD=b \sin C$$

when $\angle C$ is a right angle $AD=AC=b=b \sin 90°$

$$=b \sin C. \quad [\because \ \sin 90°=1,\ \angle C=90°$$

Hence in all three cases $AD=b \sin C$ ......(2)

From (1) and (2) $b \sin C=c \sin B$

$$\text{or } \frac{b}{\sin B}=\frac{c}{\sin C}.$$

Similarly by drawing perpendicular from C on AB, we can prove that $\frac{a}{\sin A}=\frac{b}{\sin B}$.

Hence $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$.

Note Since a, b, c are proportional to sin A, sin B, sin C, they can be replaced by sin A, sin B, sin C.

Other forms are (*i*) $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$.

(*P. U. 1917, 28*)

(*ii*) a cosec A = b cosec B = c cosec C.

(*P. U. 1918*)

Example 1. *From the sine formula deduce*

(*i*) $a=b\cos C+c\cos B$. (*P. U. 1932*)

(*ii*) $\cos A=\frac{b^2+c^2-a^2}{2bc}$ (*P. U. 1948*)

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k \text{ suppose}$$

$$\therefore \quad a=k\sin A,\ b=k\sin B,\ c=k\sin C$$

$$\begin{aligned} b\cos C+c\cos B &= k\sin B\cos C+k\sin C\cos B \\ &= k[\sin B\cos C+\cos B\sin C] \\ &= k\sin(B+C)=k\sin(180-A) \\ &= k\sin A \\ &= a. \end{aligned}$$

$$\begin{aligned} b^2+c^2-a^2 &= k^2(\sin^2 B+\sin^2 C-\sin^2 A) \\ &= k^2\{\sin^2 B+\sin(C+A)\sin(C-A)\} \\ &= k^2\{\sin^2 B+\sin B\sin(C-A)\} \\ &= k^2\sin B[\sin B+\sin(C-A)] \\ &= k^2\sin B[\sin(C+A)+\sin(C-A)] \\ &= k^2\sin B.2\sin C.\cos A \\ &= 2bc\cos A \end{aligned}$$

Hence $\cos A=\frac{b^2+c^2-a^2}{2bc}$.

Note. These formulæ will be independently proved later on.)

Example 2. *Prove that* $\sin\frac{A-B}{2}=\frac{a-b}{c}.\cos\frac{C}{2}$.

(*P. U. 1926, 1931, 1934*)

since $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin c}=k$ (suppose)

$$\therefore \quad \frac{a-b}{c}=\frac{\sin A-\sin B}{\sin C}$$

$$=\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}$$

$$=\frac{\sin\frac{C}{2}\sin\frac{A-B}{2}}{\sin\frac{C}{2}\cos\frac{C}{2}}=\frac{\sin\frac{A-B}{2}}{\cos\frac{C}{2}}$$

$$\frac{a-b}{c}\cos\frac{C}{2}=\sin\frac{A-B}{2}$$

The other forms are : (*i*) $\sin\dfrac{B-C}{2}=\dfrac{b-c}{a}\cos\dfrac{A}{2}$

(*ii*) $\sin\dfrac{C-A}{2}=\dfrac{c-a}{b}\cos\dfrac{B}{2}$

For 2nd method see Art. 10.10. Ex. 3 of this Chapter.

Example 3. *Prove that* $\cos\dfrac{B-C}{2}=\dfrac{b+c}{a}\sin\dfrac{A}{2}$

$$\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{a}{\sin A}=k$$

$$\therefore \quad \frac{b+c}{a}=\frac{\sin B+\sin C}{\sin A}$$

$$=\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$=\frac{2\cos\frac{A}{2}\cos\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}$$

$$\therefore \quad \frac{b+c}{a}\sin\frac{A}{2}=\cos\frac{B-C}{2}$$

Cor. If $b+c=2a$, *i.e*, b, a, c are in A.P.

$$\cos\frac{B-C}{2}=\frac{2a}{a}\cdot\sin\frac{A}{2}=2\sin\frac{A}{2}. \qquad (P.\ U.\ 1942)$$

Note. It can be independently proved by writing

$$\sin B+\sin C=2\sin A$$

Other Forms are :—

$$\cos\frac{C-A}{2}=\frac{c+a}{b}\sin\frac{B}{2}$$

$$\cos\frac{A-B}{4}=\frac{a+b}{c}\cdot\sin\frac{C}{2}$$

Example 4. *Prove that*

$$a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0.$$

(*P. U. 1938*)

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k \text{ suppose}$$

$$a=k\sin A,\ b=k\sin B,\ c=k\sin C$$

$$\begin{aligned}\text{Given Exp.}&=k\sin A\sin(B-C)+k\sin B\sin(C-A)+\\&\qquad k\sin C\sin(A-B\\&=k[\sin(B+C)\sin(B-C)+\sin(C+A)\\&\qquad+\sin(C-A)+\sin(A+B)\sin(A-B)]\\&=k[\sin^2B-\sin^2C+\sin^2C-\sin^2A+\sin^2A-\sin^2B]\\&=k\times0\\&=0.\end{aligned}$$

Example 5. *In a triangle ABC prove that*

$$(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^4)\cot C=0$$

(*P. U. 1944*)

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$$

$$a=k\sin A,\ b=k\sin B,\ c=k\sin C$$

$$\text{Given exp.}=k^2\left[(\sin^2B-\sin^2C)\frac{\cos A}{\sin A}+\text{two similar expressions.}\right]$$

$$=\Sigma k^2\left[\sin(B+C)\sin(B-C)\frac{\cos A}{\sin A}\right]$$

But $\sin (B+C)=\sin A$, $\cos A=-\cos (B+C)$

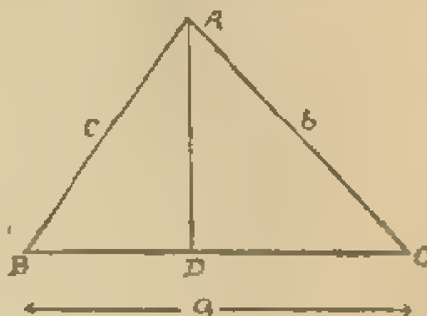
$$\text{Exp.}=-\Sigma k^2\{\cos (B+C) \sin (B-C)+\ldots+\ldots\}$$

$$=\frac{-k^2}{2}[(\sin 2B-\sin 2C+(\sin 2C-\sin 2A)+(\sin 2A-\sin 2B)]$$

$$=0.$$

10·2. Projection Formula. In any triangle ABC, prove that $a=b \cos C+c \cos B$. (*P. U. 1941, 1944-S*)

Let ABC be the $\triangle$, AD $\perp$ on BC or BC produced. Three cases arise as $\angle C$ is acute, obtuse or a right angle.



(*a*) Acute angled triangle

$$a=BC=BD+DC$$

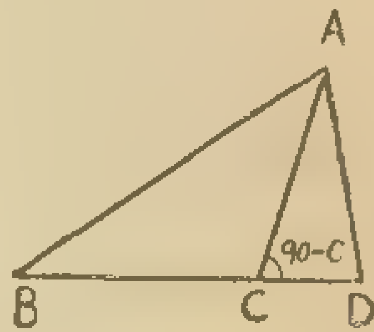
$$\cos B=\frac{BD}{AB}, \ BD=c \cos B$$

$$\cos C=\frac{DC}{AC}, \ DC=b \cos C$$

Hence $a=BC=BD+DC$

$$=c \cos B+b \cos C$$

(*b*) Obtuse angled triangle



$$a=BC=BD-CD$$

$$\cos B=\frac{BD}{AB}, \ BD=c \cos B$$

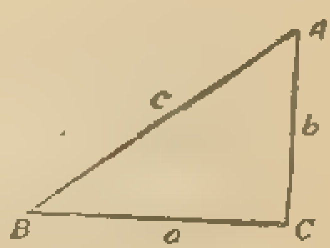
$$\cos (180-C)=\frac{CD}{AC}, \ -\cos C=\frac{CD}{b},$$

$$CD=-b \cos C$$

Hence $a=BD-BD$

$$=c \cos B+b \cos C.$$

(*c*) Right angled triangle



$$\cos B=\frac{BC}{AB}=\frac{a}{c}$$

$$a=c \cos B$$

$$=c \cos B+b \cos 90^\circ$$

$$[\because \cos 90=0]$$

$$=c \cos B+b \cos C$$

The other forms are :

$$b=a \cos C+c \cos A$$

$$c=a \cos B+b \cos A$$

(*P. U. 1928, 1938*)

Cor. Adding the three forms

$$a+b+c=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C.$$

(*P. U. 1938-S, C. U.*)

Example 1. *From the formula $c=a\cos B+b\cos A$, deduce that* $\sin(A+B)=\sin A\cos B+\cos A\sin B.$

(*P. U. 1940-S*)

$$\because \quad c=a\cos B+b\cos A$$

and $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$

$$\sin C=\sin A\cos B+\sin B\cos A$$

$$[\sin C=\sin[180-\overline{A+B}]=\sin(A+B)$$

$$\sin(A+B)=\sin A\cos B+\cos A\sin B.$$

Example 2. *From the projection formula deduce*

(I) $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}$ (*P. U. 1932 re-exam.*)

(II) $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$ (*P. U. 1939-S*)

(I) $\because\ a=b\cos C+c\cos B$, and $b=a\cos C+c\cos A$

$\therefore\ a-b\cos C-c\cos B=0$

and $a\cos C-b+c\cos A=0$

Solving for a, b, c by the method of cross multiplication

$$\frac{a}{-\cos A\cos C-\cos B}=\frac{b}{-\cos B\cos C-\cos A}$$

$$=\frac{c}{-1+\cos^2 C}$$

$$\frac{a}{\cos A\cos C+\cos B}=\frac{b}{\cos B\cos C+\cos A}$$

$$=\frac{c}{1-\cos^2 C}$$

Now $\quad \cos B=\cos[180-(A+C)]=-\cos(A+C)$

Similarly $\cos A=-\cos(B+C)$

$$\therefore\ \frac{a}{\cos A\cos C-\cos(A+C)}=\frac{b}{\cos B\cos C-\cos(B+C)}$$

$$=\frac{c}{\sin^2 C}$$

or $\dfrac{a}{\sin A}=\dfrac{b}{\sin b}=\dfrac{c}{\sin C}.$

II. $a = b\cos C + c\cos B$ ......(1)

$b = c\cos A + a\cos C$ ......(2)

$c = a\cos B + b\cos A$ ......(3)

Multiply (1) $-a$, (2) by b (3) by c and add

$$\therefore\ b^2 + c^2 - a^2 = -ab\cos C - ac\cos B + bc\cos A + ab\cos C + ac\cos B + bc\cos A$$

$$= 2bc\cos A$$

Hence $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Example 3. *In any triangle, prove that*

$b^2\sin 2C + c^2\sin 2B = 2bc\sin A$. { *A. U. 1939*, *D. U. 1941*, *P. U. 1937-S* }

L. H. S. $= b^2\sin 2c + c^2\sin 2B$

$= 2b^2\sin C\cos C + 2c^2\sin B\cos B$

$= 2b\sin C\,[b\cos C + c\cos B]$ $[\because\ c\sin B = b\sin C]$

$= 2ab\sin C$ [Projection formula]

$= 2bc\sin A$ $[\therefore\ a\sin C = \sin A]$

Otherwise

Given Exp. $= 2b^2\sin C\cos C + 2c^2\sin B\cos B$

$= 2bc\sin B\cos C + 2bc\sin C\cos B$ $[\therefore\ b\sin C = c\sin B$

$= 2bc\sin(B+C)$

$= 2bc\sin A$

Example 4. *If $a\cos^2\dfrac{C}{2} + c\cos^2\dfrac{A}{2} = \dfrac{3b}{2}$ show that the sides of the triangle are in A. P.*

$$1 + \cos C = 2\cos^2\frac{C}{2}$$

$$1 + \cos A = 2\cos^2\frac{A}{2}$$

Since $2a\cos^2\dfrac{C}{2} + 2c\cos^2\dfrac{A}{2} = 3b$

$a(1+\cos C) + c(1+\cos A) = 3b$

$a + c + (a\cos C + c\cos A) = 3b$

$a + c + b = 3b$

$a + c = 2b$

Thus the sides a, b, c are in A. P.

10·3. The Cosine Formula.

In any triangle ABC,—prove that

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \text{ or } a^2 = b^2+c^2-2bc \cos A.$$

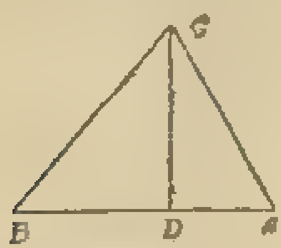
(*P. U. 1944, 1947-48, D. U. 1940*)

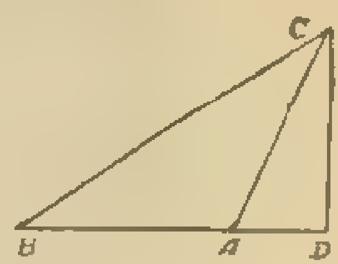
Let ABC be a triangle, $\angle A$ (say) may be acute, obtuse or a right angle.

Draw $CD \perp BA$ or BA produced.

(*a*) When $\angle A$ is acute.

From geometry $BC^2 = AB^2 + AC^2 - 2BA.AD$.





Also $\frac{AD}{AC} = \cos A \quad \therefore \quad AD = b \cos A$

$\therefore \quad a^2 = c^2 + b^2 - 2cb \cos A = b^2 + c^2 - 2bc \cos A.$

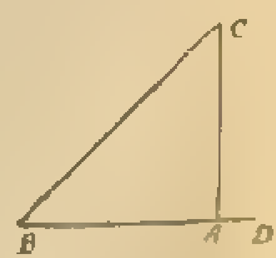
(*b*) When $\angle A$ is obtuse

From geometry $BC^2 = AC^2 + AB^2 + 2AB.AD$.

Also $\frac{AD}{AC} = \cos(180 - A) = -\cos A$

$\therefore \quad a^2 = b^2 + c^2 - 2bc \cos A$

(c) When $\angle A$ is a right $\angle$



$BC^2 = AC^2 + BA^2$

$\therefore \quad a^2 = b^2 + c^2$

$= b^2 + c^2 - 2bc \cos 90°$ [$\because$ $\cos 90° = 0$

$= b^2 + c^2 - 2bc \cos A$ [$\angle A = 90°$

Hence in all the three cases

$$a^2 = b^2 + c^2 - 2bc \cos A \text{ or } \cos A = \frac{b^2+c^2-a^2}{2bc}$$

The other forms are

(*i*) $b^2 = c^2 + a^2 - 2ca \cos B$ or $\cos B = \frac{c^2+a^2-b^2}{2ca}$

(*ii*) $c^2 = a^2 + b^2 - 2ab \cos C$ or $\cos C = \frac{a^2+b^2-c^2}{2ab}$

(*P. U. 1927*)

Alternative Method. (See figs, as above.)

When $\angle A$ is acute, $b^2 = CD^2 + AD^2$

$$a^2 = CD^2 + BD^2$$

Subtracting $b^2 - a^2 = AD^2 - BD^2$

$$= (AD+BD)(AD-BD)$$
$$= c(AD-BD)$$

$$\therefore \quad AD - BD = \frac{b^2-a^2}{c}$$

and $AD + BD = c$.

Adding $2\,AD = \dfrac{b^2-a^2}{c} + c = \dfrac{b^2+c^2-a^2}{c}$

But $\dfrac{AD}{AC} = \cos A$ or $AD = b\cos A$

$\therefore\ 2b\cos A = 2AD = \dfrac{b^2+c^2-a^2}{c}$ or $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

when $\angle A$ is obtuse, proceeding as before

$$b^2 - a^2 = AD^2 - BD^2$$
$$= (AD - BD)(AD + BD)$$
$$= -c(AD + BD)$$

$$\therefore\ (AD+CD) = -\frac{b^2-a^2}{c}$$

$$-(AD - BD) = c$$

$$\therefore \quad -2AD = \frac{b^2-a^2}{c} + c = \frac{b^2+c^2-a^2}{c}$$

But $\dfrac{AD}{AC} = \cos(180 - A) = -\cos A$

$$-AD = b\cos A$$

$\therefore\ 2b\cos A = \dfrac{b^2+c^2-a^2}{c}$ or $\cos A = \dfrac{b^2+c^2-a}{2bc}$

10'4. From the cosine formula deduce

(*i*) Sine formula (*ii*) Projection formula

(*i*) $\dfrac{a^2}{\sin^2 A} = \dfrac{a^2}{1-\cos^2 A} = \dfrac{a^2}{1 - \left[\dfrac{b^2+c^2-a^2}{2bc}\right]^2}$

$$= \frac{4a^2b^2c^2}{4b^2c^2 - (b^2+c^2-a^2)^2}$$

$$=\frac{4a^2b^2c^2}{(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)}$$

$$=\frac{4a^2b^2c^2}{[(b+c)^2-a^2][a^2-(b-c)^2]}$$

$$=\frac{4a^2b^2c^2}{(b+c+a)(b+c-a)(a-b+c)(a+b-c)}$$

$$=\frac{4a^2b^2c^2}{2s(2s-2a)(2s-2b)(2s-2c)}$$

$$=\frac{a^2b^2c^2}{4s(s-a)(s-b)(s-c)}$$

$$\frac{a}{\sin A}=\frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}$$

The symmetry of the result shows that

$\frac{b}{\sin B}$ and $\frac{c}{\sin C}$ are also equal to the same expression.

Hence the result.

(*ii*) $b\cos C+c\cos B=b\frac{a^2+b^2-c^2}{2ab}+c\frac{c^2+a^2-b^2}{2ca}$

$$=\frac{a^2+b^2-c^2+c^2+a^2-b^2}{2a}=\frac{2a^2}{2a}=a.$$

Example 2. *Prove that*

$a(b\cos C-c\cos B)=b^2-c^2$. (*P. U. 1944-S*)

$a[b\cos C-c\cos B]$

$$=ab\frac{a^2+b^2-c^2}{2ab}-ac\frac{a^2+c^2-b^2}{2ac}$$

$$=\frac{a^2+b^2-c^2}{2}-\frac{a^2+c^2-a^2}{2}$$

$$=\frac{a^2+b^2-c^2-a^2-c^2+b^2}{2}=b^2-c^2.$$

Example 3. *In any triangle ABC, prove that*

$(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0.$

(*P. U. 1944*)

I term $=(b^2-c^2)\frac{\cos A}{\sin A}$

$$=\frac{(b^2-c^2)(b^2+c^2-a^2)}{2abck}=\frac{1}{2abck}[b^4-c^4-a^2(b^2-c^2)]$$

II term$=\dfrac{1}{2abck}[c^4-a^4-b^2(c^2-a^2)]$

III term$=\dfrac{1}{2abck}[a^4-b^4-c^2(a^2-b^2)]$.

Adding. Given Exp.$=0$.

For other method see Art. I, Ex. 5.

Example 4. *If b, c, B are given, and $b<c$, show that*

$$(a_1-a_2)^2+(a_1+a_2)^2 \tan^2 B=4b^2$$

where a_1, a_2 are the two values of the third side, (*P. U., D. U.*)

$$\cos B=\frac{c^2+a^2-b^2}{2ca}$$

$a^2-2ca\cos B+c^2-b^2=0$, which is a quadratic in 'a'

Let a_1, a_2 be the roots of this equation

$\therefore\quad a_1+a_2=2c\cos B \qquad$ (sum of the roots)

$\qquad a_1a_2=c^2-b^2 \qquad$ (production of the roots)

$$\therefore\ (a_1-a_2)^2=(a_1+a_2)^2-4a_1a_2$$
$$=4c^2\cos^2 B-4(c^2-b^2)$$

$$(a_1-a_2)^2+(a_1+a_2)^2\tan^2 B$$
$$=4c^2\cos^2 B-4(c^2-b^2)+4c^2\cos^2 B\tan^2 B$$
$$=4c^2(\cos^2 B+\sin^2 B)-4c^2+4b^2$$
$$=4c^2-4c^2+4b^2$$
$$=4b^2.$$

10·5. Napier Analogy.

In any triangle, to prove that

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

(*P. U. 1923-S, '25, '29, '36, '47-S, '49*)

$$\frac{a}{\sin B}=\frac{c}{\sin C}\ \text{[Sine Formula]}$$

$$\frac{b}{c}=\frac{\sin B}{\sin C}.$$

By componendo and dividendo

$$\frac{b-c}{b+c}=\frac{\sin B-\sin C}{\sin B+\sin C}=\frac{2\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$=\frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}=\frac{\tan\frac{B-C}{2}}{\tan\left[90-\frac{A}{2}\right]}$$

$$=\frac{\tan\frac{B-C}{2}}{\cot\frac{A}{2}}$$

$$\therefore\ \tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

The other forms are :—

$$\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}.$$

$$\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}.$$

(*P. U. 1918, 1919, 1933, 1939-S., 1945*)

Note. The above result is sometimes written as

$$\frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}=\frac{b-c}{b+c}.$$

10·6 In any triangle, prove that

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

(*P. U. 1937-S, '50, '42, '46, '47-S, '48*)

Or Express $\sin\frac{A}{2}$ in terms of ths sides of the triangle.

$$\cos A=1-2\sin^2\frac{A}{2},$$

$$2\sin^2\frac{A}{2}=1-\cos A$$

$$2\sin^2\frac{A}{2}=\frac{1}{2}(1-\cos A)$$

$$=\frac{1}{2}\left(1-\frac{b^2+c^2-a^2}{2bc}\right.\qquad\text{[Cos Formula]}$$

$$=\frac{1}{2}\left(\frac{2bc-b^2-c^2+a^2}{2bc}\right)=\frac{a^2-(b-c)^2}{4bc}$$

$$=\frac{(a-b+c)(a+b-c)}{4bc}$$

Now $a+b+c=2s$, $a+b+c-2b=2s-2b$

$$a-b+c=2(s-b)$$

$$a+b-c=2(s-c)$$

$$\sin^2\frac{A}{2}=\frac{2(s-b)2(s-c)}{4bc}=\frac{(s-b)(s-c)}{bc}$$

$$\therefore \quad \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$\because \quad A<180,\ \frac{A}{2}<90.$

Hence the positive sign.

The other forms are : $\sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ac}}$

$$\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$$

Example 1. *In any triangle ABC, prove that*

$$2\left(a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}\right)=c+a-b. \quad (P.\ U.\ 1942)$$

$$\text{L. H. S.}=2\left(a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}\right)$$

$$=2a\frac{(s-a)(s-b)}{ab}+2c\frac{(s-b)(s-c)}{bc}$$

$$=\frac{2(s-b)}{b}(s-a+s-c)=\frac{2s-2b}{b}(2s-a-c)$$

$$=\frac{(a+c+b-2b)}{b}(a+b+c-a-c)=(c+a-b).$$

2nd Method.

$$2a\sin^2\frac{C}{2}+2c\sin^2\frac{A}{2}$$

$$=a(1-\cos C)+c(1-\cos A)$$

$$=a+c-(a\cos C+c\cos A)=a+c-b.$$

10·7. In any triangle, to prove that

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}.$$

(P. U. 1924, '27-S. '45, '47, '49)

Or, to express $\cos\frac{A}{2}$ in terms of the sides of the triangle.

$$2\cos^2\frac{A}{2}=1+\cos A.$$

$$\cos^2\frac{A}{2}=\frac{1}{2}\,[1+\cos A]$$

$$=\frac{1}{2}\left[1+\frac{b^2+c^2-a^2}{2bc}\right]\quad[\because \text{ cos Formula}]$$

$$=\frac{1}{2}\left[\frac{2bc+b^2+c^2-a^2}{2bc}\right]=\frac{1}{2}\,\frac{(b+c)^2-a^2}{2bc}$$

$$=\frac{(b+c+a)(b+c-a)}{4bc}.$$

Now $\quad a+b+c=2s$

$\therefore\quad b+c-a=b+c+a-2a=2s-2a=2(s-a).$

$\therefore\quad \cos^2\frac{A}{2}=\frac{2s.2(s-a)}{4bc}=\frac{s(s-a)}{bc}$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$$

$\because\quad A<180,\ \frac{A}{2}<90.$ Hence the positive sign.

The other forms are : $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}.$

$$\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}.$$

Example. *Prove that*

$$c^2=(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2}.$$

(P. U. 1941 '50.)

R. H. S. $=(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\left(1-\cos^2\frac{C}{2}\right)$

$$\left[\because\ \sin^2\frac{C}{2}=1-\cos^2\frac{C}{2}\right]$$

$$=(a+b)^2-[(a+b)^2-(a-b)^2]\cos^2\frac{C}{2}$$

$$=(a+b)^2-4ab\,\frac{s(s-c)}{ab}$$

$$=(a+b)^2-2s(2s-2c)$$

$$=(a+b)^2-(a+b+c)(a+b-c)$$

$$=(a+b)^2-(a+b)^2+c^2$$

$$=c^2.$$

2nd Method. Change $\cos\frac{C}{2}$ into $\sin\frac{C}{2}$

and apply $\sin\frac{C}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$

3rd Method. $2\cos^2\frac{C}{2}=1+\cos C$

$$2\sin^2\frac{C}{2}=1-\cos C$$

R. H. S. $=(a-b)^2\frac{(1+\cos C)}{2}+(a+b)^2\frac{(1-\cos C)}{2}$

$$=\tfrac{1}{2}[(a-b)^2+(a+b)^2]-\tfrac{1}{2}\{(a+b)^2-(a-b)^2\}\cos C$$

$$=\tfrac{1}{2}[2(a^2+b^2)]-2ab\,\frac{a^2+b^2-c^2}{2ab}$$

$$=a^2+b^2-a^2-b^2+c^2$$

$$=c^2.$$

The other forms are :

$$a^2=(b-c)^2\cos^2\frac{A}{2}+(b+c)^2\sin^2\frac{A}{2}$$

$$b^2=(c-a)^2\cos^2\frac{B}{2}+(c+a)^2\sin^2\frac{B}{2}.$$

10·8. In any triangle ABC, prove that

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

(*P. U.* 1916, 24, 27-S, 30, 43, 50)

$$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

The other forms are :—

$$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Example 1. $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}$

$+\tan\frac{C}{2}\tan\frac{A}{2}=1.$ (A. U. 1935)

Given Exp.

$$=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\cdot\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}+\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$$

$$\times\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}+\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\times\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$=\frac{s-c}{s}+\frac{s-a}{s}+\frac{s-b}{s}=\frac{3s-(a+b+c)}{s}=\frac{3s-2s}{s}=\frac{s}{s}=1.$$

2nd Method. See Chapter VI, Art. 6, Ex. 2.

3rd Method.

$$\tan\left(\frac{A}{2}+\frac{B}{2}+\frac{C}{2}\right)$$

$$=\frac{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}-\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}-\tan\frac{B}{2}\tan\frac{C}{2}-\tan\frac{C}{2}\tan\frac{A}{2}}$$

$$\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90, \text{ and } \tan 90=\infty$$

Hence the denominator is zero.

Example 2. *Prove that*

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}.$$

(*P. U. 1942*)

L.H.S.

$$=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$=\frac{\sqrt{s(s-a)}+\sqrt{s(s-b)}+\sqrt{s(s-c)}}{\sqrt{(s-a)(s-b)(s-c)}}$$

$$=\frac{\sqrt{s(s-a+s-b+s-c)}}{\text{Den.}}=\frac{\sqrt{s[3s-(a+b+c)]}}{\text{Den.}}$$

$$=\frac{\sqrt{s.s}}{\sqrt{(s-a)(s-b)(s-c)}}$$

$$\text{R.H.S.}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\times\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$=\frac{\sqrt{s.s}}{\sqrt{(s-a)(s-b)(s-c)}}$$

Hence L.H.S=R.H.S.

2nd Method. Consult example 6 on identities.

10·9. In any triangle ABC, to prove that

$$\sin A=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

Now $\sin A=2\sin\frac{A}{2}\cos\frac{A}{2}$

$$=2\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{s(s-a)}{bc}}$$

$$=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

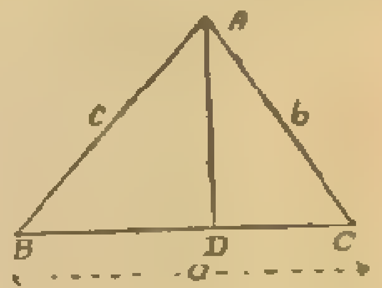
The other forms are

$$\sin B=\frac{2}{ac}\sqrt{s(s-a)(s-b)(s-c)}$$

$$\sin C=\frac{2}{ab}\sqrt{s(s-a)(s-b)(s-c)}$$

Cor. 1. $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=\frac{2}{abc}\sqrt{s(s-a)(s-b)(s-c)}$

Cor. 2. Area of the triangle. (*P. U.*)



$\Delta=\frac{1}{2}$ base $\times$ height

$=\frac{1}{2}a\,AD$

$=\frac{1}{2}ac\sin B$

$=\frac{1}{2}ab\sin C$

$=\frac{1}{2}bc\sin A$

$=\frac{1}{2}bc.\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

$=\sqrt{s(s-a)(s-b)(s-c)}$

This is known as Hero's Formula.

10·10. Solved Examples.

Example 1. *If* $a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{3b}{2}$, *then the sides of the triangle are in A.P.*

$$\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}},\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$$

$$\therefore\ a\frac{s(s-c)}{ab}+c\frac{s(s-a)}{bc}=\frac{3b}{2}$$

$$\text{or}\quad \frac{2s(s-c)}{b}+\frac{2s(s-a)}{b}=3b$$

$$\frac{2s}{b}[s-c+s-a]=3b$$

$$\frac{2s}{b}.b=3b$$

$$a+b+c=3b$$

$$a+c=2b$$

Hence a, b, c are in A. P.

2nd Method. See Art. 10·2, Ex. 4. Chapter X.

Example 2. *Prove that in any triangle*

$$(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}=0.$$

(*P.U. 1941-S*)

1st Method.

$$\text{L.H.S.}=(\sin B-\sin C)\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\text{two similar terms.}$$

$$=2\cos\frac{B+C}{2}\sin\frac{B-C}{2}\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\ldots\ldots+\ldots\ldots$$

$$=2\sin\frac{A}{2}\sin\frac{B-C}{2}\cdot\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}+\ldots\ldots+\ldots\ldots$$

$$=2\sin\frac{B-C}{2}\cdot\cos\left[90-\frac{B+C}{2}\right]+\ldots\ldots+\ldots\ldots$$

$$=2\sin\frac{B-C}{2}\sin\frac{B+C}{2}+\ldots\ldots+\ldots\ldots$$

$$=\cos C-\cos B+\cos A-\cos C+\cos B-\cos A$$

$$=0.$$

2nd Method.

$$\text{L.H.S.}=(b-c)\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+(c-a)\sqrt{\frac{s(s-b)}{(s-a)(s-b)}}+(a-b)\sqrt{\frac{s(s-c)}{(s-b)(s-c)}}$$

$$=\frac{(b-c)\sqrt{s(s-a)}+(c-a)\sqrt{s(s-b)}+(a-b)\sqrt{s(s-c)}}{\sqrt{(s-a)(s-b)(s-c)}}$$

$$\text{Num.}=\frac{\sqrt{s}}{2}[(b-c)(2s-2a)+(c-a)(2s-2b)+(a-b)(2s-2c)]$$

$$=\frac{\sqrt{s}}{2}[(b-c)(b+c-a)+(c-a)(c+a-b)+(a-b)(a+b-c)]$$

$$=\frac{\sqrt{s}}{2}[b^2-c^2-ab+ac+c^2-a^2+bc+ab+a^2-b^2-ac+bc]$$

$$=0.$$

Example 3. *In any triangle ABC, prove that*

$$\sin\frac{B-C}{2}=\frac{b-c}{a}\cos\frac{A}{2} \qquad \text{(P. U.)}$$

$$\sin\frac{B-C}{2}=\sin\frac{B}{2}\cos\frac{C}{2}-\cos\frac{B}{2}\sin\frac{C}{2}$$

$$=\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{s(s-c)}{ab}}-\sqrt{\frac{s(s-b)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$=\frac{s-c}{a}\sqrt{\frac{s(s-a)}{bc}}-\frac{s-b}{a}\sqrt{\frac{s(s-a)}{bc}}$$

$$=\left[\frac{s-c}{a}-\frac{s-b}{a}\right]\sqrt{\frac{s(s-a)}{bc}}$$

$$=\frac{b-c}{a}\cos\frac{A}{2}.$$

For 2nd method see Art. 10·1, Ex. 2 of this chapter.

Example 4. *Prove that*

$$a\sin\left(\frac{A}{2}+B\right)=(b+c)\sin\frac{A}{2}$$

(*P. U. 1943 F.Sc. Ag. 1944*)

$$\frac{b}{a}=\frac{\sin B}{\sin A},\quad \frac{c}{a}=\frac{\sin C}{\sin A}$$

Adding $\dfrac{b+c}{a}=\dfrac{\sin B+\sin C}{\sin A}=\dfrac{\sin B+\sin[180-(A+B)]}{\sin A}$

$$=\frac{\sin B+\sin(A+B)}{\sin A}=\frac{2\sin\left(\frac{A}{2}+B\right)\cos\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$=\frac{\sin\left(\frac{A}{2}+B\right)}{\sin\frac{A}{2}}$$

$$\therefore\quad a\sin\left(\frac{A}{2}+B\right)=(b+c)\sin\frac{A}{2}.$$

2nd Method.

First prove $\cos\dfrac{B-C}{2}=\dfrac{b+c}{a}\sin\dfrac{A}{2}$ See Art. 1, Ex. 3

Now we have to get rid of C

$$C=180-(A+B)$$

$$\cos\frac{B-C}{2}=\cos\frac{B-[180-(A+B)]}{2}$$

$$=\cos\frac{B-180+A+B}{2}=\cos\left(\frac{A}{2}+B-90\right)$$

$$=\cos\left[90-\left(\frac{A}{2}+B\right)\right] \quad \therefore \quad \cos(-\theta)=\cos\theta$$

$$=\sin\left(\frac{A}{2}+B\right)$$

$$\therefore \quad a\sin\left(\frac{A}{2}+B\right)=(b+c)\sin\frac{A}{2}.$$

Example 5. *If the sides of a triangle are in* **A. P.**, *prove that so are the co tangents of half the angles.*

(*P. U. 1943-S, Nagpur 1940*)

1st Method. $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are in A. P.

If $\cot\frac{A}{2}+\cot\frac{C}{2}=2\cot\frac{B}{2}$.

If $\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=2\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}$

Multiplying both sides by $\sqrt{(s-a)(s-b)(s-c)}$ and divide by $\sqrt{s}$.

If $\quad (s-a)+(s-c)=2(s-b)$

If $\quad 2s-(a+c)=2s-2b$

$\quad a+c=2b$ which is given.

2nd Method. a, b, c are in A. P.

$$2b=a+c$$

$$2\sin B=\sin A+\sin C \qquad \text{(Note Art. 1)}$$

$$\sin B=\sin A-\sin B+\sin C$$

$$2\sin\frac{B}{2}\cos\frac{B}{2}=4\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} \qquad \text{(Identity)}$$

$$\sin\frac{B}{2}\cos\frac{B}{2}=2\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$\sin\frac{B}{2}\cos\left(90-\frac{A+C}{2}\right)=2\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$\sin\frac{B}{2}\sin\frac{A+C}{2}=2\sin\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$\frac{\sin\frac{A+C}{2}}{\sin\frac{A}{2}\sin\frac{C}{2}}=\frac{2\cos\frac{B}{2}}{\sin\frac{B}{2}}$$

$$\frac{\sin\frac{A}{2}\cos\frac{C}{2}+\sin\frac{C}{2}\cos\frac{A}{2}}{\sin\frac{A}{2}\sin\frac{C}{2}}=2\cot\frac{B}{2}$$

$$\cot\frac{A}{2}+\cot\frac{C}{2}=2\cot\frac{B}{2}$$

Hence $\cot\frac{A}{2}$, $\cot\frac{B}{2}$, $\cot\frac{C}{2}$ are in A. P.

Example 6. *If the sides of a triangle are in A.P., prove that*

$$\cot\frac{A}{2}\cot\frac{C}{2}=3.$$

1st Method. $2b=a+c$

$$2\sin B=\sin A+\sin C$$

$$4\sin\frac{B}{2}\cos\frac{B}{2}=2\sin\frac{A+C}{2}\cos\frac{A-C}{2}$$

$$=2\sin\left(90-\frac{B}{2}\right)\cos\frac{A-C}{2}=2\cos\frac{B}{2}\cos\frac{A-C}{2}$$

$$\therefore \quad 2\sin\frac{B}{2}=\cos\frac{A-C}{2}$$

$$2\sin\left(90-\frac{A+C}{2}\right)=\cos\frac{A-C}{2}$$

$$2\cos\frac{A+C}{2}=\cos\frac{A-C}{2}$$

$$2\left(\cos\frac{A}{2}\cos\frac{C}{2}-\sin\frac{A}{2}\sin\frac{C}{2}\right)$$

$$=\cos\frac{A}{2}\cos\frac{C}{2}+\sin\frac{A}{2}\sin\frac{C}{2}$$

$$\cos\frac{A}{2}\cos\frac{C}{2}=3\sin\frac{A}{2}\sin\frac{C}{2}$$

$$\cot\frac{A}{2}\cot\frac{C}{2}=3.$$

2nd Method. We have to prove

$$\cot\frac{A}{2}\cot\frac{C}{2}=3$$

or $\sqrt{\dfrac{s(s-a)}{(s-b)(s-c)}}\times\sqrt{\dfrac{s(s-c)}{(s-a)(s-b)}}=3$ or $\dfrac{s}{s-b}=3$

$$s=3s-3b$$

$$3b=2s=a+b+c$$

$2b=a+c$ which is given.

Steps may be reversed.

Example 7. *If a, b, c are in H.P., prove that $\sin^2\frac{A}{2}$, $\sin^2\frac{B}{2}$ and $\sin^2\frac{C}{2}$ are also in H.P.* (P.U. 1937-S)

We know that $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$ and similar expressions for $\sin\dfrac{B}{2}$ and $\sin\dfrac{C}{2}$.

$\sin^2\dfrac{A}{2}$, $\sin^2\dfrac{B}{2}$, $\sin^2\dfrac{C}{2}$ are in H. P., if

$$\frac{bc}{(s-b)(s-c)}+\frac{ab}{(s-a)(s-b)}=2\,\frac{ca}{(s-c)(s-b)}$$

if $\quad \dfrac{s-a}{a}+\dfrac{s-c}{c}=2\,\dfrac{s-b}{b}.$

$\left\{\text{We have multiplied by } \dfrac{(s-a)(s-b)(s-c)}{abc}\right\}$

If $$\frac{s}{a}-1+\frac{s}{c}-1=\frac{2s}{b}-2$$

$$\frac{1}{a}+\frac{1}{c}=\frac{2}{b}.$$

If a, b, c are in H. P.

2nd Msthod. We have to prove that

$$\frac{1}{\sin^2\frac{A}{2}}+\frac{1}{\sin^2\frac{C}{2}}=\frac{2}{\sin^2\frac{B}{2}}.$$

This is true if $\operatorname{cosec}^2\frac{A}{2}+\operatorname{cosec}^2\frac{C}{2}=2\operatorname{cosec}^2\frac{B}{2}$

i.e., if $1+\cot^2\frac{A}{2}+1+\cot^2\frac{C}{2}=2\left(1+\cot^2\frac{B}{2}\right)$

i.e., if $\dfrac{s(s-a)}{(s-b)(s-c)}+\dfrac{s(s-c)}{(s-a)(s-b)}=\dfrac{2s(s-b)}{(s-a)(s-c)}$

i.e., if $(s-a)^2+(s-c)^2=2(s-b)^2$

i.e., if $(s-a)^2-(s-b)^2=(s-b)^2-(s-c)^2$

i.e., if $c(b-a)=a(c-b)$

i.e., if $$\frac{c}{a}=\frac{c-b}{b-a}$$

i.e., if a, b, c are in H. P.

Example 8. *The sides of a triangle are 7, 3 and 5; show that the greatest angle is 120°.*

1st Method. $\sin a=\dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$

Here $2s=7+3+5=15$; $\therefore$ $s=\frac{15}{2}$.

$\therefore$ $s-a=\frac{15}{2}-7=\frac{1}{2}$, $s-b=\frac{15}{2}-3=\frac{9}{2}$, $s-c=\frac{15}{2}-5=\frac{5}{2}$

$\therefore$ $$\sin A=\frac{2}{3\cdot5}\sqrt{\frac{1}{2}\cdot\frac{9}{2}\cdot\frac{5}{2}\cdot\frac{15}{2}}=\frac{2}{3\cdot5}\cdot\frac{3\cdot5}{2\cdot2}\sqrt{3}=\frac{\sqrt{3}}{2}$$

$\therefore$ A = 60° or 120°. The first value being inadmissible, is rejected.

2nd Method. $\cos A=\dfrac{b^2+c^2-a^2}{2bc}=\dfrac{9+25-49}{2.3.5}=\dfrac{-15}{30}=-\dfrac{1}{2}$

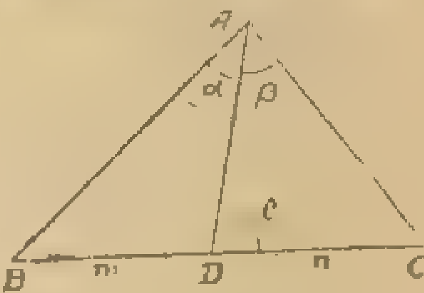
$\therefore$ $A=120°$

Notes. The question can also be solved by considering the formulae for

$$\cos\frac{A}{2},\ \sin\frac{A}{2},\ \tan\frac{A}{2},\ \cot\frac{A}{2}\ \text{etc.}$$

Example 9. *Two Important Theorems.*

If D divides the base BC of the $\triangle ABC$ such that $BD : DC :: m : n$ and $\angle ADC = \theta$ and $\angle BAD = \alpha$ and $\angle DAC = \beta$ then



(*i*) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$

(*ii*) $(m+n)\cot\theta = n\cot B - m\cot C$

$$\frac{AD}{m} = \frac{\sin B}{\sin\alpha} = \frac{\sin(\theta-\alpha)}{\sin\alpha}$$

$$= \sin\theta\cot\alpha - \cos\theta$$

$$\frac{AD}{n} = \frac{\sin C}{\sin\beta} = \frac{\sin(180-\theta-\beta)}{\sin\beta} = \frac{\sin(\theta+\beta)}{\sin\beta}$$

$$= \sin\theta\cot\beta + \cos\theta.$$

Equating the values of AD, we have

$$m(\sin\theta\cot\alpha - \cos\theta) = n(\sin\theta\cot\beta + \cos\theta)$$

or $\qquad (m+n)\cos\theta = m\sin\theta\cot\alpha - n\sin\theta\cot\beta$

$\therefore \qquad (m+n)\cot\theta = m\cot\alpha - n\cot\beta$

Again $\qquad \dfrac{AD}{m} = \dfrac{\sin B}{\sin\alpha} = \dfrac{\sin B}{\sin(\theta-B)}$

$$= \frac{\sin B}{\sin\theta\cos B - \cos\theta\sin\beta} \qquad ...(i)$$

$$\frac{AD}{n} = \frac{\sin C}{\sin\beta} = \frac{\sin C}{\sin(180-\theta-C)} = \frac{\sin C}{\sin(\theta+c)}$$

$$= \frac{\sin C}{\sin\theta\cos C + \cos\theta\sin C}.$$

Equating the values of AD, we get

$$\frac{m\sin B}{\sin\theta\cos B - \cos\theta\sin B} = \frac{n\sin C}{\sin\theta\cos C + \cos\theta\sin C}$$

or $$\frac{m}{\cot B - \cot\theta} = \frac{n}{\cot C + \cot\theta}$$

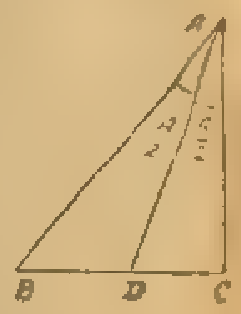
or $m(\cot C + \cot\theta) = n(\cot B - \cot\theta)$

i.e., $\qquad (m+n)\cot\theta = n\cot B - m\cot C \qquad ...(ii)$

N.B. The above theorems are very useful in Statics.

Example 10. *The bisector of the angle A meets the base at D, show that*

$$AD=\frac{2bc}{b+c}\cos\frac{A}{2}.$$



Since AD is the bisector of $\angle BAC$

$$\therefore \quad \frac{BD}{DC}=\frac{c}{b}$$

$$\therefore \quad \frac{\Delta ABD}{\Delta ACD}=\frac{BD}{DC}=\frac{c}{b}$$

$$\therefore \quad \frac{\Delta ABD+ACD}{\Delta ACD}=\frac{b+c}{b}$$

or $$\frac{\Delta ABC}{\Delta ADC}=\frac{b+c}{b}$$

$$\therefore \quad \frac{\frac{1}{2}bc\sin A}{\frac{1}{2}AD.b\sin\frac{A}{2}}=\frac{b+c}{b} \quad \text{[Art. 9 Cor. 2]}$$

or $$2c\cos\frac{A}{2}=AD\,\frac{b+c}{b}$$

or $$AD=\frac{2bc}{b+c}\cos\frac{A}{2}.$$

EXAMPLES

In any triangle ABC, prove that

1. $a(\sin B-\sin C)+b(\sin C-\sin A)+c(\sin A-\sin B)=0.$

2. $2(bc\cos A+ca\cos B+ab\cos C)=a^2+b^2+c^2.$

3. $$\frac{\cos A}{c\cos B+b\cos C}+\frac{\cos B}{a\cos C+c\cos A}+\frac{\cos C}{b\cos A+a\cos B}=\frac{a^2+b^2+c^2}{2abc}.$$

[**Hint**: Apply cosine and projection formulae.]

4. $$\frac{b\sec B+c\sec C}{\tan B+\tan C}=\frac{c\sec C+a\sec A}{\tan C+\tan A}=\frac{a\sec A+b\sec B}{\tan A+\tan B}$$ ***(P. U. 1926, D. U. 1944)***

5. $a(\cos B+\cos C)=2(b+c)\sin^2\frac{A}{2}.$ *(Nagpur, 1943)*

[**Hint:** $\dfrac{b+c}{a}=\dfrac{\sin B+\sin C}{\sin A}$

or $\qquad b=c\cos A+a\cos C$

$\qquad c=a\cos B+b\cos A$

Add $\quad b+c=a(\cos B+\cos C)+(b+c)\cos A$

$a(\cos B+\cos C)=(b+c)(1-\cos A)=(b+c)2\sin^2\dfrac{A}{2}$.

6. $a(\cos C-\cos B)=2(b-c)\cos^2\dfrac{A}{2}$. (*P. U. 1940-S*)

7. $s\tan\dfrac{A}{2}=(s-b)\cot\dfrac{C}{2}=(s-c)\cot\dfrac{B}{2}$.

[**Hint:** Put down the value of the t ratios of half the ∠s. in terms of sides.]

8. $(s-a)\tan\dfrac{A}{2}=(s-b)\tan\dfrac{B}{2}=(s-c)\tan\dfrac{C}{2}$.

9. $a\cos A+b\cos B+c\cos C=2a\sin B\sin C$
$=2b\sin A\sin C$
$=2c\sin A\sin B$. (*P. U. 1940*)

[**Hint:** L.H.S.$=\dfrac{K}{2}[\sin 2A+\sin B+\sin 2C]=\dfrac{K}{2}.\ 4\sin A$ $\sin B\sin C$ from identities etc.]

10. $(a+b+c)\sin\dfrac{A}{2}=2a\cos\dfrac{B}{2}\cos\dfrac{C}{2}$.

[**Solution**

$$\frac{a+b+c}{k}=\sin A+\sin B+\sin C$$
$$=2\sin\frac{A}{2}\cos\frac{A}{2}+2\sin\frac{B+C}{2}\cos\frac{B-C}{2}$$
$$=2\sin\frac{A}{2}\cos\frac{A}{2}+2\sin\left(90-\frac{A}{2}\right)\cos\frac{B-C}{2}$$
$$=2\sin\frac{A}{2}\cos\frac{A}{2}+2\cos\frac{A}{2}\cos\frac{B-C}{2}$$
$$=2\cos\frac{A}{2}\left[\sin\frac{A}{2}+\cos\frac{B-C}{2}\right]$$
$$=2\cos\frac{A}{2}\left[\sin\left\{90-\frac{B+C}{2}\right\}+\cos\frac{B-C}{2}\right]$$

$$=2\cos\frac{A}{2}\left[\cos\frac{B+C}{2}+\cos\frac{B-C}{2}\right]$$

$$=2\cos\frac{A}{2}\cdot 2\cos\frac{B}{2}\cos\frac{C}{2}$$

$$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\therefore\quad (a+b+c)\sin\frac{A}{2}=k2\sin\frac{A}{2}\cos\frac{A}{2}\cdot 2\cos\frac{B}{2}\cos\frac{C}{2}$$

$$=2k\sin A\cos\frac{B}{2}\cos\frac{C}{2}$$

$$=2a\cos\frac{B}{2}\cos\frac{C}{2}\Big].$$

11. $(b+c-a)\sin\frac{A}{2}=2a\sin\frac{B}{2}\sin\frac{C}{2}$.

12. $a\sin A=b\sin B+c\sin(A-B)$. *(C. U. 1931)*

13. $a^2+b^2+c^2=2(ab\cos C+bc\cos A+ca\cos B)$.

[Hint: Add the three forms of Cos Formula.]

14. $\dfrac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}=\dfrac{a+b}{a-b}$. *(P. U. 1952)*

[Hint: R. H. S. $=\dfrac{\sin A+\sin B}{\sin A-\sin B}$ etc.]

15. $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$. *(C. U. 1919)*

[L. H. S. $=(b\cos A+a\cos B)+(c\cos A+a\cos C)$ $+(c\cos B+b\cos C)$]

16. $\dfrac{\sin(A-B)}{\sin(A+B)}=\dfrac{a^2-b^2}{c^2}$.

17. $a(b\cos C-c\cos B)=b^2-c^2$. *(C. U.)*

18. $(a+c)\tan\frac{B}{2}+(a-c)\cot\frac{B}{2}=2c\cot C$.

19. $c^2=(a-b)^2\cos^2\frac{C}{2}+(a+b)^2\sin^2\frac{C}{2}$. *(P. U. 1941, 50)*

[See solved exercise after Art. 10·7]

20. $a\cos\frac{B-C}{2}=(b+c)\sin\frac{A}{2}$.

21. $\sin\frac{A-B}{2}=\frac{a-b}{c}\cos\frac{C}{2}$ *(P. U. 1931, 34)*

22. $(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0.$

23. $a^2+b^2+c^2=2(bc\cos A+ca\cos B+ab\cos C).$

24. $2\left(a\sin^2\frac{C}{2}+c\sin^2\frac{A}{2}\right)=c+a-b.$ (*P. U. 1942*)

25. $\frac{b-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^4}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0.$

(*P. U. 1912*)

26. $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc.$

[Sol. Exp. $=\Sigma a^3\cos(B-C)=k\Sigma a^2\sin A\cos(B-C)$

$=k\Sigma a^2\sin(B+C)\sin(B-C)$

$=\frac{k}{2}\Sigma a^2(\sin 2B+\sin 2C)=k\Sigma a^2(\sin B\cos B+\sin C\cos C)$

$=\Sigma a^2(b\cos B+c\cos C)=\Sigma ab(b\cos A+a\cos B)=3abc.$

27. $a^3\sin(B-C)+b^3\sin(C-A)+c^3\sin(A-B)=0.$

28. $(a+b+c)(\cos A+\cos B+\cos C)$

$=2\left(a\cos^2\frac{A}{2}+b\cos^2\frac{B}{2}+c\cos^2\frac{C}{2}\right).$

29. $\frac{\cos B}{\cos C}=\frac{c-b\cos A}{b-c\cos A}.$

30. $a\cos(B-C)=b\cos B+c\cos C.$ (*C. U.*)

31. $(b^2+c^2-a^2)\tan A=(c^2+a^2-b^2)\tan B$

$=(a^2+b^2-c^2)\tan C.$

32. $(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}+(a-b)\cot\frac{C}{2}=0.$

33. $\frac{b^2-c^2}{a^2}\sin 2A+\frac{c^2-a^2}{b^2}\sin 2B+\frac{a^2-b^2}{c^2}\sin 2C=0.$

(*C. U. 1912*)

34. $\tan\left(\frac{A}{2}+B\right)=\frac{c+b}{c-b}\tan\frac{A}{2}.$

35. $a(s-a)\sec^2\frac{A}{2}=b(s-b)\sec^2\frac{B}{2}=c(s-c)\sec^2\frac{C}{2}.$

36. $a+b+c=2a\cos\frac{B}{2}\cos\frac{C}{2}\operatorname{cosec}\frac{A}{2}.$

37. $\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{2}=\frac{a^2+b^2+c^2}{2abc}.$

38. $\cot \frac{B}{2} \cot \frac{C}{2} = 2$ if $b+c=3a$.

39. $b(b+c-a)(1-\cos A) = a(a+c-b)(1-\cos B)$.

40. $\frac{a \sin (B-C)}{b^2-c^2} = \frac{b \sin (C-A)}{c^2-a^2} = \frac{c \sin (A-B)}{a^2-b^2}$.

41. $(a+b+c)\left(\tan \frac{A}{2} + \tan \frac{B}{2}\right) = 2 \cot \frac{C}{2}$.

42. $a \sin \frac{A}{2} \sin \frac{B-C}{2} + b \sin \frac{B}{2} \sin \frac{C-A}{2}$

$$+ c \sin \frac{C}{2} \sin \frac{A-B}{2} = 0.$$

MISCELLANEOUS EXERCISES ON CHAPTER X

1. If $a \cos A = b \cos B$, prove that the $\triangle$ is either isosceles or right angled. *(P. U. 1939-S)*

[Hint : $a \frac{b^2+c^2-a^2}{2bc} = b \times \frac{c^2+a^2-b^2}{2ac}$

On simplification $c^2(a^2-b^2) = (a^2-b^2) \times (a^2+b^2)$

$\therefore$ $a^2-b^2=0$ $\therefore$ the $\triangle$ is isosceles

or $c^2=a^2+b^2$ or the $\triangle$ is rt $\angle$d.]

2. If $\sin^2 A = \sin^2 B + \sin^2 C$, prove that the $\triangle$ is rt. $\angle$d.

Hint : Put $\sin A = Ka$ etc.]

3. If $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, prove that the $\triangle$ is equilateral.

[Hint : Put $a = K \sin A$ etc.

$\therefore$ $\tan A = \tan B = \tan C$ etc.

4. If $(a+b+c)(c+b-a) = 3bc$, prove that $\angle A = 60^\circ$.

Hint : The given condition gives

$$b^2+c^2+2bc-a^2=3bc$$

or $$\frac{b^2+c^2-a^2}{2bc} = \frac{1}{2} = \cos A].$$

5. If ABC be a $\triangle$ and another $\triangle$ be formed whose sides are c, $(a+b) \sin \frac{C}{2}$, $(a-b) \cos \frac{C}{2}$, prove that it is rt. $\angle$d.

Hint : Prove that $c^2 = (a+b)^2 \sin^2 \frac{C}{2} + (a-b)^2 \cos^2 \frac{C}{2}$ vide solved ex. Art. 10·7].

6. If AD be the $\perp$ from A on BC, prove that)

$$AD=\frac{b^2 \sin C+c^2 \sin B}{b+c}.$$

7. If a, b, c are in A. P., that

$$\cos A+2\cos B+\cos C=2.$$

8. If $2\cos B=\frac{\sin A}{\sin C}$, prove that the triangle is isosceles. (*C. U.*)

9. If the bisectors of the base angles of a triangle are equal. prove that it is isosceles,

10. If the sides of a triahgle are 7. 8 and 13, show that the greater angle is 120°.

11. If the median AD divides the angle A into angles α and β show that (*i*) $\cot\alpha-\cot\beta=\cot B-\cot C$.

(*ii*) $\cot\alpha \qquad =2\cot A+\cot B$.

12. If $\cot\frac{A}{2}=\frac{b+c}{2}$ show that the $\triangle$ is rt, $\angle$d. (*P. U. 1937*)

13. Prove that in any triangle

$$(a+b-c)\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)=2c\cot\frac{C}{2}$$

14. Prove that in any triangle

$$(i)\quad \frac{\cos^2\frac{A}{2}}{a}+\frac{\cos^2\frac{B}{2}}{b}=\frac{\cos^2\frac{C}{2}}{c}=\frac{s^2}{abc}.$$

$$(ii)\quad \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=\frac{4\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}}{a+b+c}$$

(*P. U. 1928*)

15. Show that

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{a+b+c}{a+b-c}.\cot\frac{C}{2}$$ (*D. U.*)

CHAPTER XI

LOGARITHMS

11·1. It will be necessary now for the student to become acquainted with the nature and use of logarithms, as it is a powerful instrument in mathematics and is a great labour saving device. All the elementary properties of logarithms can be deduced from the theory of indices in Algebra. The chief laws of indices are given below :—

(1) $a^0=1$
(2) $a^1=a$
(3) $a^x.a^y=a^{x+y}$
(4) $a^x \div a^y=a^{x-y}$
(5) $(a^x)^y=a^{xy}$.

11·2. Def. The logarithm of any number to a given *base* is the index of the power to which the base must be raised in order to equal the given number. Thus if $a^x=N$, x is called the logarithm of N to the base a. The logarithm of N to base a is usually written as $\log_a N$, (read as log N to the base a) so that the same meaning is expressed by the two equations

$$a^x=N\,; \qquad x=\log_a N.$$

Substituting the value of x from the latter in the former we get the identity

$$N=a^{\log_a N}$$

which is sometimes useful.

EXAMPLES

(1) Since $3^4=81$, $\log_3 81=4$.

(2) Since $10^3=1000$, $\log_{10}1000=3$.

(3) Since $(64)^{\frac{1}{6}}=2$, $\log_{64}2=\frac{1}{6}$.

(4) Since $5^{-3}=\frac{1}{125}$, $\log_5 \frac{1}{125}=-3$.

(5) Since $4^{-\cdot 5}=\cdot 5$, $\log_4 \cdot 5=-\cdot 5$.

(6) Since $16^{\frac{5}{4}}=32$, $\log_{16}32=\frac{5}{4}$.

(7) Find $\log_9 27$

Let $\log_9 27 = x$, then $9^x = 27$

$\therefore \quad 3^{2x} = 3^{3}$

$2x = 3 \quad \text{or} \quad x = \frac{3}{2}.$

Hence $\log_9 27 = \frac{3}{2} = 1{\cdot}5$

(8) Find the logarithm of $\frac{1}{3}$ to the base 27. (*P. U.*)

Let $\log_{27} \frac{1}{3} = x \quad \therefore \quad 27^x = \frac{1}{3}$

$\therefore \quad (3^3)^x = (3)^{-1} \quad \text{or } 3x = -1 \quad \therefore \quad x = -\frac{1}{3}.$

Since the base a in the general exponential function $a^x = N$, is always taken to be a positive number other than unity, the number N can never be negative. Hence the equation $\log_a N = x$, is only valid when N is positive or in other words the logarithm of any negative number is not real. If $a > 1$, then $a^{-\infty} = 0$, so that $\log_a 0 = -\infty$, Similarly, if $a < 1$, $a^{\infty} = 0$

$\therefore \quad \log_a 0 = \infty$

11·3. Two Important special cases of logarithms.

(*i*) $\log_a 1 = 0$ [In words: The logarithms of 1 to any base $= 0$]

(*ii*) $\log_a a = 1$ in words: The logarithms of any base (with respect) to itself $= 1$]

Proof. (*i*) Since $a^0 = 1$: for all values of a $\therefore$ by definition 0 is the logarithm of 1 to the base a, *i.e.*, $\log_a 1 = 0$.

(*ii*) Since $a = a$, $\therefore$ by definition 1 is the logarithm of a to the base a, *i.e.*, $\log_a a = 1$.

FOUR FORMULAE OF LOGARITHMS

I. Product Formula

11·31. To prove that $\log_a mn = \log_a m + \log_a n$.

[In words: The logarithm of the product of two or more numbers is equal to the sum of the logarithms of the factors.]

Proof. Let $\log_a m = x$ and $\log_a n = y$.

$\therefore$ By definition of logarithms

$$m = a^x \text{ and } n = a^y$$

$$\therefore \quad m \cdot n = a^x \cdot a^y = a^{x+y},$$

Again by definition. $mn=a^{x+y}$ gives

$$\log_a mn = x+y$$
$$= \log a^m + \log_a{}^n.$$

Cor. $\log_a m\, n.p\ldots\ldots = \log_a m + \log_a n + \log_a p + \ldots\ldots$

II. Quotient Formula

12.32. To prove that $\log_a \dfrac{m}{n} = \log_a{}^m - \log_a{}^n$.

[In words : The logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator.]

Proof. Let $\log_a{}^m = x$ and $\log_a{}^n = y$.

∴ By definition of logarithms

$$m=a^x \text{ and } n=a^y$$
$$\therefore \frac{m}{n}=a^{x-y}.$$

Now $\dfrac{m}{n}=a^{x-y}$ gives, by definition

$$\log_a \frac{m}{n} = x-y$$
$$=\log_a{}^m - \log_a{}^n.$$

Cor. $\log_a \dfrac{m.n}{p.q} = \log_a{}^m+\log_a{}^n-\log_a p-\log_a q.$

III. Power Formula

13.33. To prove that $\log m^n = n.\log_a m$.

[In words : The logarithm of a number raised to a power is equal to the index of the power into the logarithm of the number.]

Proof. Put $\log_a m = x$ $\quad\therefore\quad m=a^x$.

$\therefore \quad (m)^n=(a^x)^n=a^{nx}$

$\therefore \log_a(m^n)=nx$

$\qquad\qquad = p.\log_a m,$

Example. *Express* $\log_x \dfrac{m^2.\sqrt[3]{n}}{\sqrt[5]{p^4}}$ *in terms of*

$\log_a{}^m, \log_a{}^n, \log_a p.$

Sol. $\log_a \dfrac{m^2.\sqrt[3]{n}}{\sqrt[5]{p^4}} = \log \dfrac{m^2.n^{\frac{1}{3}}}{p^{\frac{4}{5}}}$

$= \log_a m^2 + \log_a n^{\frac{1}{3}} - \log_a p^{\frac{4}{5}}$ [Art. 13'31 and 13'32]

$= 2\log_a m + \frac{1}{3}\log_a n - \frac{4}{5}\log_a p.$ [Art. 13'32]

IV. Formula for Change of Base

11'34. To prove that

(*i*) $\log_a m = \log_b m \times \log_a b$

(*ii*) $\log_b m = \dfrac{\log_a m}{\log_a b}.$

[In words: Given the logarithm of a number N to the base *a*, to find the logarithm of the same number N to the base *b*.]

Prof. (*i*) Put $\log_b m = x$ and $\log_a b = y$

then, by def. $m = b^x$ and $b = a^y$.

$\therefore\ m = (b)^x = (a^y)^x = a^{xy}.$

[Note this step; we are aiming at eliminating *b*, because on the L. H. S. there is no *b*.]

$\therefore\ \log_a m = xy = \log_b m . \log_a b.$

(*ii*) This result follows from result (*i*), by dividing both sides by $\log_a b$.

However this result can also be proved independently.

Put $\log_a m = x$ and $\log_a b = y$.

$\therefore$ By def. $m = a^x$ and $b = a^y$.

Note here we require to eliminate '*a*' because on the L. H. S. there is no '*a*'.

$\therefore\ m = a^x$

$= \left[(b)^{\frac{1}{y}}\right]^x \left[\because \text{ from } b = a^y \text{ we get } a = (b)^{\frac{1}{y}}\right]$

$= b^{\frac{x}{y}}$

$\therefore\ \log_b m = \dfrac{x}{y} = \dfrac{\log_a m}{\log_a b}.$

Caution. The student should note to avoid the following mistakes

(*i*) $\log_a(m \pm n) = \log_a m \pm \log_a n.$

(*ii*) $\log_a 2n \quad = 2\log_a n.$

11·4. Two Systems of Logarithms.

I. Common Logarithms

Def. When logarithms are taken to the base 10, they are called common Logarithms; they are used in all practical calculations when no base is expressed, the base 10 is implied therein.

II. Natural Logarithms

Def. Logarithms to the base 'e' are called Natural Logarithms, e being the sum of the infinite series

$$1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\ldots\ldots=2{\cdot}71828$$

This system of logarithms is also known as Naperian logarithms.

We shall in this book confine ourselves to common logarithms and whenever we write logarithms we mean common logarithms.

11·5. Characteristic and Mantissa.

Def. The integral part of the logarithm of a number (after expressing the decimal part as positive) is called the **Characteristic** and the positive decimal part is called the **Mantissa**.

We shall explain this point as follows. We know that the logarithm of a number is not always integral. Thus since $10^3=1{,}000$ and $10^4=10{,}000$ the logarithm of a number lying between 1,000 and 10,000 lies between 3 and 4, and is, therefore, equal to $3+a$ positive proper fraction. Similarly, ·0045 lies between ·001 and ·01, *i.e.*, between 10^{-3} and 10^{-2}, the logarithm of ·0045 lies between -3 and -2, *i.e.*, it is equal to $-3+a$ positive proper fraction. Thus we see that the logarithm of a number consists, in general, of two parts, the integral part, called the **characteristic** which may be positive or negative and the fractional or decimal part, called the **mantissa**, **which must always be positive.**

Example. *The logarithm of a number is* (i) *3·4010* (ii) *$-1{\cdot}9050$, find the characteristic and mantissa.*

(i) 3·4010. Here the decimal part is already positive, hence the characteristic (integral part) is ·4010.

Note. It should be noted clearly that mantissa is always positive.

(*ii*) $-1{\cdot}9050$. Here the decimal $(-{\cdot}9050)$ is negotive. To make it positive, add and subtract one from the number and write the -1 with the integral part and $+1$ with the decimal part.

Thus $-1{\cdot}9050=-1-{\cdot}9050$

$=(-1-1)+(1-{\cdot}9050)$ add and subtract 1

$=-2+{\cdot}0950.$

$\therefore$ The characteristic here is -2 and mantissa is ${\cdot}0950$.

Notation. The result $-2+0{\cdot}950$ is, for shortness, written as $\bar{2}{\cdot}0950$, and is read as 2 bar decimal 0950. The horizontal bar placed over 2 denotes that the integral part is alone negative and that the mantissa (or decimal part) is always positive.

11 6. Rules for finding the chrracteristic.

Rule I. The characteristic of the logarithm of a number greater than unity, is an integer, less by one than the number of digits in the integral part of the number.

Proof Let N be a number >1, having n digits its integral part.

Since $10^0=1$, $10^1=10$, $10^2=100$, $10^3=1000$, $10^4=10{,}000,\ldots$ it follows that a number with two digits in its integral part lies between 10^1 and 10^2 and number having three digits lies between 10^2 and 10^3 and so on, for example, 345 lies between 100 and 1,000, *i e*, between 10^2 and 10^3. Hence a number with n digits lies between 10^{n-1} and 10^n. Let N be any number with n digits, then $10^{n-1}<N<10^n$ and therefore $\log 10^{n-1}<\log N<\log 10^n$ *i.e.*, $n-1<\log N<n$, *i e*, $\log N=(n-1)+$a proper fraction

Hence the characteristic of $\log N=n-1$.

Rule II. The characteristic of the logarithm of a number less than one is negative and is a number greater by one than the number of zeros between the decimal point and the first significant digit.

Proof. Let N be a numcer <1, having n zeros immediately after the decimal point.

Since $10^{-1}={\cdot}1$, $10^{-2}={\cdot}01$ and so on, it follows that a number with one cypher immediately after the decimal

point such as ·035, being greater than ·01 and less than ·1 lies between 10^{-2} and 10^{-1}, similarly a number with two cyphers after the decimal point lies between 10^{-3} and 10^{-2} and so on. In general, a number N, having n zeros before the first significant digit lies between $10^{-(n+1)}$ and 10^{-n}. Its logarithm, therefore, lies, between $-(n+1)$ and $-n$, *i.e.*, $\log N=-(n+1)+$ a proper fraction. Hence the characteristic is $-(n+1)$.

Example 1. *Find the characteristic of the logarithms of* (*i*) *6592.* (*ii*) *6·592.* (*iii*) *65·92.*

Sol. (*i*) $6592>1$ and the number of digits in the integral part $=4$.

$\therefore$ the characteristic of the logarithm $=4-1=3$.

(*ii*) 6·592. Here the number of digits in the integral part $=1$.

$\therefore$ the characteristic of the logarithm $=1-1=0$.

(*iii*) 65·92. Here the number of digits in the integral part $=2$.

$\therefore$ the characteristic of the logarithm $=2-1=1$.

Example 2. *Find the characteristic of the logarithms of* (*i*) *·2035.* (*ii*) *·02035.* (*iii*) *·002035.*

Sol. (*i*) $·2035<1$. Here there is no zero immediately after the decimal part, *i.e.*, the number of zeros immediately after the decimal part $=0$ $\therefore$ the characteristic of the logarithm

$$=-(0+1)=-1=\bar{1}.$$

(*ii*) ·02035. Here the number of zeros immediately after the decimal point $=1$, $\therefore$ the characteristic of the logarithm

$$=-(1+1)=-2=\bar{2}.$$

(*iii*) ·002035. Here the number of zeros immediately after the decimal point $=2$, $\therefore$ the characteristic of the logarithm

$$=-(2+1)=-3=\bar{3}.$$

11·7. Rule for the Mantissa. The mantissae of the logarithms of all numbers consisting of the same digits in the same order, are the same.

Proof. Let P and Q be two numbers differing in the position of the decimal point only. Then

$$\frac{P}{Q}=10^m \text{ where } m \text{ is an integer positive or negative.}$$

[Explanation : If $P=4321$ and $Q=\cdot004321$

then $\dfrac{P}{Q}=\dfrac{4321}{\cdot004321}=\dfrac{4321}{\dfrac{4321}{1000000}}=1000000=10^6$

or if $P=\cdot876$ and $Q=876$, we get

$\dfrac{P}{Q}=\dfrac{\cdot876}{876}=\dfrac{876}{1000}\times\dfrac{1}{876}=\dfrac{1}{1000}=10^{-3}$]

$\therefore\quad \log P-\log Q=\log 10^m=m\log 10$

$\qquad\qquad =m \quad [\because \log 10=\log_{10}10=1]$

$\therefore$ The logorithms of P and Q differ by an integer and hence the decimal part remains unaffected and $\therefore$ the mantissæ are the same.

EXAMPLES

Example 1. *If log 7645=3·8834, write down the logarithms of 7·645, 07645, ·0007645.* (*P. U.*)

Sol. The numbers differ only in the position of the decimal point only. $\therefore$ The mantissa of all the numbers is the same as that of log 7645.

$\therefore\quad$ Mantissa$=\cdot8834$

Characteristic of $7\cdot645=1-1=0.$

" " $\cdot07645=-(1+1)=-2=\bar{2}$

" " $\cdot007645=-(2+1)=-3=\bar{3}$

$\therefore$ (*i*) log $7\cdot645=\cdot8834$

(*ii*) log $\cdot07645=\bar{2}\cdot8834$

(*iii*) log $\cdot007645=\bar{3}\cdot8834$

Example 2. *Given log 2=·30103, find*

(*i*) *the number of digits in 2^{64}.*

(*ii*) *the position of the first significant figure in 2^{-37}.*

(*P. U. 1941*)

Sol. $\log 2^{64}=64\times\log 2.$

$\qquad =64\times\cdot30103=19\cdot26592.$

$\therefore$ the characteristic of the logarithm of 2^{64} is 19. This is one less than the number of digits in (the integral part) 2^{64}.

(*ii*) $\log 2^{-37}=-37\log 2=-37\times\cdot30103=-11\cdot13911.$

$\qquad =\bar{12}\cdot86189.$

$\therefore$ The characteristic of the logarithm of $2^{-37} = -12$. This is numerically one more than the number of zeros immediately after the decimal point in 2^{-37}.

$\therefore$ in 2^{-37} there are 11 zeros immediately after the decimal point *ie*, the first significant figure in the 12th place of decimals.

Example 3. *Given that* $\log 3 = \cdot 4771$, $\log 7 = \cdot 8451$ *and* $\log 11 = 1\cdot 0414$, *solve the equation* $3^x \times 7^{2x+1} = 11^{x+5}$.

(*P. U. 1943-S*)

Sol. Taking logarithms we have

$$\log 3^x + \log 7^{2x+1} = \log 11^{x+5}.$$

or $x \log 3 + (2x+1) \log 7 = (x+5) \log 11$.

$\therefore$ $x(\log 3 + 2\log 7 - \log 11) = 5\log 11 - \log 7$.

$$\therefore \quad x = \frac{5\log 11 - \log 7}{\log 3 + 2\log 7 - \log 11}$$

$$= \frac{5\cdot 2070 - \cdot 8451}{\cdot 4771 + 1\cdot 6902 - 1\cdot 0414}$$

$$= \frac{4\cdot 3619}{1\cdot 1259} = 3\cdot 8 \text{ nearly.}$$

Note. There are two main advantages of using 10 as the base of a system of logarithms :

(*i*) The characteristic of the logarithms of a number can be determined by inspection.

(*ii*) The mantissa of the logarithms of all numbers consisting of the same digits in the same order (*i e.*, of numbers differing only in the position of the decimal point) are the same.

EXERCISE XI (A)

1. If $\log_a 64 = 5$, find a.
   [**Hint :** $a^5 = 64 = 2^5$ $\therefore$ $a = 2$],

2. Prove that $a^{\log_a x} = x$.
   [**Sol.** Put $\log_a x = n$ $\therefore$ $a^n = x$...(1)
   $\therefore$ Putting the value of n in (1)
   $a^{\log_a x} = x$.]

3. What are the characteristics of the logarithms of
   (*a*) 123·4, 5·027 13·53, 1006.
   (*b*) ·0345, ·1023 ·0013, ·00057.

4. Find the values of

(*i*) $\log_4 256$. (*ii*) $\log_2 16$. (*iii*) $\log_{81} 243$.

5. Given that $\log 2 = \cdot 3010$, find the values of

(*i*) $\log \cdot 0005$. (*ii*) $\log (6\cdot 4)^{-3}$.

6. Given $\log 3 = \cdot 4771$, find the number of digits in 3^{62} and the position of the first significant figure in 3^{-65}.

7. Given $\log 2 = \cdot 3010$ and $\log 3 = \cdot 4771$, solve the equations (*i*) $2^x . 3^{2x} = 100$. (*ii*) $5^{1-x} = 6^{x-2}$.

8. Given $\log 2 = \cdot 3010$, $\log 3 = \cdot 4771$ and $\log 7 = \cdot 8451$, find the value of x and y from the equations:

(*i*) $2^x = 7^y$. (*ii*) $7^{x+y} \times 3^{2x+y} = 9$.

and $2^{y-1} = 7^{x+1}$ $3^{x-y} = 3^x + 2^{x-2y}$.

[Hint: By taking logarithms, we get

$$x \log 2 = y \log 7$$

$$\text{and } (y-1) \log 2 = (x+1) \log 7 \text{ etc.}]$$

9. Given $\log 2 = \cdot 3010$ and $\log 3 = \cdot 4771$, calculate to two decimal places the values of (*i*) $\log_8 27$ (*ii*) $\log_3 10$.

$\left[\textbf{Hint: } \text{Use } \log_b x = \dfrac{\log_{10} x}{\log_{10} b} = \dfrac{\log x}{\log b}\right]$.

SECTION II

11·8. Mathematical Tables.

In a book of Four-Figure Tables (printed at the end of the book), the mantissae of the logarithms of all numbers consisting of four significant figures are given correct to four places of decimals, with the decimal point dropped (for convenience of printing); the characteristics can be written down by inspection and are therefore omitted.

How to read the tables.

The extreme left hand column gives the first two significant figures of the number. The next ten columns, headed by 0, 1, 2......, 9, correspond to the third figure of the number. The small columns to the right, called the difference columns, are similarly headed by 1, 2, 3,......9 and these figures correspond to the fourth significant number. The tables may be used to find the mantissa of a given number.

Example 1. *Find log 46·73.*

1. Characteristic is $2-1=1$.

2. To find mantissa.

(*i*) Removing the decimal point, we get 4673. The first two significant figures are 46, the third figure is 7 and the fourth is 3.

(*ii*) In the table of Logarithms [pages, *ii*, *iii* at the end of the book], we look up for 46, down the extreme left-hand columns.

(*iii*) In the horizontal row beginning with 46 and under the column headed by 7, we find the number 6693 at the intersection.

(*iv*) In continuation of the horizontal row and under the 'Mean Differences' column on the right, headed by 3, we find the number 3 at the intersection.

(*v*) Adding 3 to 6993, we get 6696.

∴ the mantissa = 6696.

∴ From (1) and (8) we get log 46.73 = 1·6696.

Example 2. *Find log 7.*

(*i*) Here the characteristic is $1-1=0$.

(*ii*) The mantissa of 7 is the same as mantissa of 7000. [∴ it should be noted that if a number contains less than four (significant) figures, we add zeros to the right of the number until it contains four figures.]

Now the mantissa, from the tables, of 7000 = ·8551 [as in solved example (1), look up 70 in the first left hand column and in the horizontal row beginning with 70 and under the column headed by 0, we find the number 8451 at the intersection; as there is no column on the right headed by 0 under the 'Mean Differences', we add nothing for the fourth figure.]

∴ log 7 = ·8451.

11.81. Tables of anti-logarithms. Def. If $\log_a N = x$, then N is called the anti-logarithm of x to the base a. The integral part of x or the characteristic of x, indicates only the position of the decimal point of N. The digits constituting the number N are obtained by finding the mantissa only in the tables.

Examples. *Find the number whose logarithm is 2·7125.*

Let x be the number, so that $\log x = 2·7128$. In finding x, we shall leave the characteristic 2 for the present and take the mantissa only, *i.e.*, ·7125. See the anti-logarithm table, run

down the first column till ·71 is reached. Then in the same horizontal line under the column headed by 2 is the number 5152. This number corresponds to ·712. Again under the difference column headed by 8 and in the same horizontal row as ·71 is the number 10. This is to be added to 5152 and the result, ·5162, is the number corresponding to 7128. Hence the significant digits of x are 5162, and since the characteristic of log x is 2,

$$\therefore \quad x=516\cdot2,$$

i.e., anti log $2\cdot718=517\cdot2$.

11·9. Logarithms as an instrument of Simplification.

Example 1. *Find the cube root of ·002.*

$$\text{Let} \quad x=(\cdot002)^{\frac{1}{3}}$$
$$\therefore \quad \log x=\tfrac{1}{3}\log(\cdot002)$$
$$=\tfrac{1}{3}[\bar{3}\cdot3010]$$
$$=1[\bar{1}\cdot0103]$$
$$\therefore \quad x=\text{anti-log }[\bar{1}\cdot0103]$$
$$=\cdot1024. \text{ [from anti logarithmic tables.]}$$

Example 2. *Find the value of* $\frac{(438)^3\sqrt{\cdot056}}{(388)^4}$ *as accurately as you can.*

Sol. Let $x=\frac{(438)^3\sqrt{\cdot056}}{(388)^4}$

$$\therefore \quad \log x=3\log 438+\tfrac{1}{2}\log\cdot056-4\log 388.$$
$$=3\times2\cdot6415+\tfrac{1}{2}\times\bar{2}\cdot7415-4\times2\cdot5888.$$
$$=7\cdot9245+\bar{1}\cdot3741-10\cdot3552.$$
$$=7\cdot9235-1+\cdot3741-10\cdot3552.$$
$$=-3\cdot0566=-4+1-\cdot0566.$$
$$=\bar{4}\cdot9436.$$
$$\therefore \quad x=\text{anti log }\bar{4}\cdot9436.$$
$$=\cdot0008782.$$

Example 2. *Evaluate:* $\left\{\frac{\sqrt[3]{1\cdot025}-\sqrt[7]{\cdot2732}}{(5\cdot73)^2}\right\}^{\frac{2}{3}}$

$$\text{Let} \quad x=\left\{\frac{\sqrt[3]{1\cdot025}\times\sqrt[7]{\cdot2732}}{(5\cdot73)^2}\right\}^{\frac{2}{3}}$$

$$\therefore \quad \log x = \left\{ \frac{2}{3} \log \frac{\sqrt[3]{1{\cdot}025} \times \sqrt[7]{{\cdot}2732}}{(5{\cdot}73)^4} \right\}$$

$$= \tfrac{2}{3} \left\{ \tfrac{1}{3} \log 1{\cdot}025 + \tfrac{1}{7} \log {\cdot}2732 - 2 \log 5{\cdot}73 \right\}$$

$$= \tfrac{2}{3} \left\{ \tfrac{1}{3} \times {\cdot}0107 + \tfrac{1}{7} [\bar{1}{\cdot}4365]^* - 2 \times {\cdot}7582 \right\}$$

$$= \tfrac{2}{3} \left\{ {\cdot}00356 + \tfrac{1}{7} [-7 + 6{\cdot}4365] - 1{\cdot}5164 \right\}$$

$$= \tfrac{2}{3} \left\{ {\cdot}00356 + \bar{1}{\cdot}9195 - 1{\cdot}5164 \right\}$$

$$= \tfrac{2}{3} \left\{ \bar{2}{\cdot}40666 \right\}$$

$$= \tfrac{1}{3} \times \bar{4}{\cdot}81332.^*$$

$$= \tfrac{1}{3} [\bar{6} + 2{\cdot}81332]$$

$$= -2 + {\cdot}9378$$

$$= \bar{2}{\cdot}9378$$

$$= {\cdot}08666 \text{ from anti-logarithm tables.}$$

EXERCISE XI (B)

1. Find the logarithms of
(*i*) ·0286. (*ii*) 5467. (*P. U.*)

2. Evaluate (*i*) $\log_2{}^4$, (*ii*) $\log_3{}^{10}$, (*iii*) $\log_{03}{}^2$. (*P. U. 1943*)

3. Find the anti-loearithms of
(*i*) 2·8249. (*ii*) 2·5832. (*iii*) 3·5147. (*P. U.*)

4. (*i*) Find the fifth root of (·003).
(*ii*) Find the seventh root of ·03457.

5. Give log 2 = ·3010300, log 4844544 = 6·6852530, find the value of

$$(6{\cdot}4)^{\frac{1}{10}} \times (\sqrt[4]{256})^3 \div \sqrt{80}. \qquad (P.\ U.\ 1944)$$

6. Find the value of (*i*) $\sqrt{\dfrac{x^{\frac{1}{5}} \times 11^{\frac{1}{3}}}{\sqrt{74} \times (62)^{\frac{1}{5}}}}$

(*ii*) $\dfrac{(6{\cdot}45)^6 \sqrt[8]{{\cdot}00034}}{(9{\cdot}37)^2 \times \sqrt[4]{8{\cdot}93}}$,

7. Find the value of
$\dfrac{(435)^3 \times \sqrt{{\cdot}056}}{(380)^4}$ as accurately as you can.

(* *Note these steps carefully.* $\bar{1}$ is not divisible by 7. ∴ we take the next higher number divisible by 7, which is 7 in this. ∴ we write $\frac{1}{7}[\bar{1}{\cdot}4365]$ as $\frac{1}{7}[-7+6{\cdot}4365]$.

And again $\bar{4}$ is not divisible by 3 ∴ we take the next higher number divisible by 3 ; that number is 6 ∴ we write $\frac{1}{3}[\bar{4}{\cdot}81332]$ as $\frac{1}{3}[-6+2{\cdot}81332]$.

11·10. Tables of Natural Trigonometric ratios. These tables show three distinct columns: (*i*) the column for degrees, (*ii*) the columns which divide a degree at intervals of 6′ minutes), and (*iii*) the Mean Difference columns for an increase of 1, 2, 3, 4 or 5 minutes in the angle.

Example 1. *Find* $\sin 70°38'$.

In the tables of natural sines [pages *vi*, *vii* at the end of the book], we look up for 70° in the column of degrees.

In the horizontal row beginning with 70°, and under the column headed by 36′, we find the number 9432.

In the continuation of the horizontal row and under the Mean Differences columns on the right, headed by 2, we find the number 2 at the intersection.

Adding 2 to 9432 we get 9434. Prefixing the decimal point we get

$$\sin 70°38' = {\cdot}9434.$$

Example 2. *Find* $\cos 28°17'$.

From the tables of Natural Cosines [pages *viii* and *ix* at the end of the book] we find

$$\cos 28°12' = {\cdot}8813$$
$$\text{Mean Difference for } 5' = \quad 7 \text{ subtract}$$
$$\therefore \quad \cos 28°17' = 8806$$

Another Method. $\cos 28°17' = \cos(90 - 61°43')$
$= \sin 61°43'$, which from the tables of natural sines $= {\cdot}8806.$

Note 1. As the angle increases from 0° to 90, the ratios sine, tangent, (and secant) increase and the co-ratios cosine, cotangent (and cosecant) decrease. Hence with the increase of the angle the numbers in the Mean Difference columns (on the right) of the tables of Natural Sines and Natural Tangents are to be added, and the numbers in the Mean Differences columns of the tables of Natural Cosines and Natural Cotangents are to be subtracted.

Note 2. For the cosines and cotangents it is found convenient in practice first to express the cosine and cotangent as a sine or tangent respectively, by using the Complementary Formulæ as $\cos(90 = \theta) = \sin\theta$ and $\cot(90 - \theta) = \tan\theta$.

Example 3. *Find the angle whose sine is ·5055.*

Sol. From the tables of Natural Sines [*page xi, vii* at the end], we find that the nearest number to 5055 but less than it is 5045 which occurs in the horizontal row beginning with 30° and under the column headed by 18′.

Now the Mean Difference between 5055 and 5045 is 10.

In continuation of the horizontal row, we find that the number 10 occurs under the Mean Differences column headed by 4′, at the intersection.

Adding 4′ to 30°18′, we get the required angle = 30°22′.

11·11. Tables of Logarithms of Trignometric ratios. These tables are used exactly the same way as the tables of Natural T-ratios. They also give (*i*) the logarithms of the trigonometric ratios of all angles from 0 to 90° at intervals of 6′ (*ii*) the Mean Differences for an increase of 1, 2, 3, 4 or 5 minutes in the angle.

The method of using the tables is illustrated by the following examples.

Example. *Find log sin 64° 26′.*

The first column contains degrees. Look for the row containing 64°. Look for the column headed by 24′. At their intersection we read the number 9551, which means that the mantissa of log sin 64°24′ is ·9551, the characteristic, $\bar{1}$, being shown only in the column headed by 0. Now we have to find the difference for two minutes. In the same row as 64° and under the difference column headed by 2′, we get the number 1 which means ·0001 and is to be added to $\bar{1}$·9551.

$$\text{Hence } \log \sin 65°26' = \bar{1}\cdot 9551 + \cdot 0001$$
$$= \bar{1}\cdot 9552.$$

Note 1. The numbers given in the difference columns are to be subtracted in the case of natural and logarithmic cosines, cotangents and cosecants. In all other cases they are to be added. The reason is that cosine, contangent and cosecant decrease as the angle increases from 0° to 90°.

Note 2. In order to avoid the occurrence of negative quantities in the tables, sometimes 10 is added to the logarithm of every circular function, before registering it in the tables;

the logarithm so increased is called the tabular logarithm, and is usually denoted by the letter L; thus L sin A means the Tabular logarithm of sine of A and it is equal to the real logarithm of the sine of A increased by 10.

11 12. The Principle of Proportional Parts. If we are required to find the logarithm of a given number which is not given in the tables or the number corresponding to a given logarithm not contained in the tables, we apply the principle of proportional parts, which states that the increase in the logarithm of a number is proportional to the increase in the number itsself. Similarly if we are required to find the trigonometrical ratio of an angle which is not contained in the tables or the angle corresponding to given trigonometrical ratios we apply the fact that the change in the trigonometrical ratio of an angle is proportional to a small change in the angle itself. We shall illustrate this principle by a few examples.

Example 1. *Find the logarithm of 34567.*

From the tables we obtain

$$\log 34560 = 4{\cdot}5386.$$

$$\log 34570 = 4{\cdot}5387.$$

If the difference between the number is 10, the difference between the logarithm is ·0001. Therefore if the difference between the numbers be 7, that between the logarithm will be ·00007. Hence

$$\log 34567 = 4{\cdot}53867.$$

Example 2. *Find log sin 24° 18′ 30″.*

From the tables we obtain,

$$\log \sin 24°18' = \bar{1}{\cdot}6144.$$

and $$\log \sin 24°19' = \bar{1}{\cdot}6147.$$

If the difference between the angles be 1′, the difference between their logarithmic sines in ·0003. Therefore if the difference between the angles be 30″, that between their logarithmic sines will be $\frac{{\cdot}0003}{2}$, *i.e.*, ·00015. Hence

$$\log \sin 24°18'30'' = \bar{1}{\cdot}6144 + {\cdot}00015$$
$$= \bar{1}{\cdot}61455.$$

Example 3. *Given log sin $x=\bar{1}\cdot91825$; find x.*

From the tables we obtain that

$$\bar{1}\cdot9182=\log\sin 55^\circ\ 55'$$
and
$$\bar{1}\cdot9183=\log\sin 55^\circ\ 56'.$$

Now if the difference between the logarithmic sines is $\cdot0001$, that between the angles is $1'$. Therefore if the difference between the logarithmic sines be $\cdot00005$ that between the angles will be $30''$. Hence

$$\bar{1}\cdot91825=\log\sin 55^\circ 55' 30''.$$
i.e.,
$$x=55^\circ 55' 30''.$$

Example 4. *Given log cos $x=\bar{1}\cdot2195$; find x.*

The number greater than $\bar{1}\cdot2195$ and nearest in the logarithmic cosine tables is $\bar{1}\cdot2221$ and we find that $\log\cos 80^\circ 24'=\bar{1}\cdot2221$. We have to count for a difference of 26, which cannot be found in the difference columns. But we find a difference of 23 under $3'$ and difference of 30 under $4'$. Hence

$$\log\cos 80^\circ 27'=\bar{1}\cdot2198$$
and
$$\log\cos 80^\circ 28'=\bar{1}\cdot2191.$$

If the difference between the logarithmic cosines is $\cdot0007$, that between the angles is $60''$. Therefore if the difference between the logarithmic cosines be $\cdot0003$, that between the angles will be $\dfrac{60''\times 3}{7}=26''$ nearly. Hence

$$\bar{1}\cdot2195=\log\cos(80^\circ 27'+26'')$$
$$=\log\cos 80^\circ 27' 26''.$$

EXERCISES XI (C)

1. From the tables find the value of
(*i*) $\sin 15^\circ 34'$. (*ii*) $\cos 47^\circ 39'$. (*iii*) $\tan 65^\circ 44'$.

2. Find to the nearest minute the angles whose tangents are $\frac{3}{4}$ and $\frac{2}{5}$. (*P.U. 1935*)

3. Evaluate
(*i*) $\log\sin 43^\circ\ 13'$. (*ii*) $\log\tan 18^\circ\ 26'$.
(*iii*) $\log\cos 72^\circ\ 37'$.

4. Evaluate (*i*) $\angle$ sin 39
(*ii*) $\angle$ tan 54.

5. Given $\log 3{\cdot}141 = {\cdot}4970$
and $\log 3{\cdot}142 = {\cdot}4972$, find $\log 3{\cdot}1416$.

6. Find the value of $\sin 29°37'42''$. (*P. U. 1944-S*)

[**Hint**: From tables $\sin 29°37' = {\cdot}4942$.
$\sin 29°38' = {\cdot}4944$.

$\therefore$ For a difference of $1'$ in the angle, the difference in the sine $= {\cdot}4942 = {\cdot}0002$.

$\therefore$ By Art. 11·12 (principle of proportional part) for a diff. of $42''$ in the angle, the diff. in the sine $= {\cdot}0002 \times \frac{42}{60} = {\cdot}00014$,

$\therefore$ $\sin 29°37'42'' = {\cdot}4942 + {\cdot}00014$ [we add ·00014 because $\sin 29°37'42'' > \sin 29°37'$]
$= {\cdot}49434$
$= {\cdot}4943$ nearly.

7. Find the number whose cosine is ·7280843. (*P. U. 1944-S*)

8. Find (*i*) $\log \sin 21°3'20''$. (*ii*) $\log \cos 84°36'14''$.

MISCELLANEOUS SOLVED EXAMPLES

1. If a, b, c be in G. P., show that $\log_a x$ $\log_b x$, $\log_c x$ are in H. P. (*P. U. 1931*)

$$a \cdot c = b^2.$$

Take logarithms of both sides to base x, then

$$\log_x ac = \log_x b^2.$$

$\therefore$ $\log_x c + \log_x a = 2 \log_x b$.

Hence $\log_x a$, $\log_x b$, $\log_x c$ are in A. P.

But $\log_a x$, $\log_b x$, $\log_c x$ are their reciprocals, since $\log_x a \times \log_a x = 1$ etc.

Hence $\log_a x$, $\log_b x$, $\log_c x$ are in H. P.

Since a, b, c are in G. P.

2. The Post Office 5-year cash certificates for Rs. 500 are obtainable at an issue price of Rs. 440 10*a*. Find the rate per cent of compound interest. (*P. U. 1940*)

Let x be the rate per cent. per annum of compound interest.

Then a sum of Re. 1 will amount to Rs. $\left(1+\frac{x}{100}\right)$ in one year and to Rs. $\left(1+\frac{x}{100}\right)^5$ in 5 years. The sum of Rs. 440 10*a*, *i.e.*, Rs. $\frac{3525}{8}$ will amount to Rs. $\frac{3525}{8}\left(1+\frac{x}{100}\right)^5$ in 5 years.

[$\because$ $A=P\left(1+\frac{R}{100}\right)^T$. But this sum should be equal to Rs. 500. Hence we have the equation

$$\frac{3525}{8}\left(1+\frac{x}{100}\right)^5=500.$$

$$\therefore\quad 1+\frac{x}{100}=\left(\frac{500\times 8}{3525}\right)^{\frac{1}{5}}.$$

$$\therefore\quad \log\left(1+\frac{x}{100}\right)=\tfrac{1}{5}\{\log 500+\log 8-\log 3525\}$$
$$=\tfrac{1}{5}\{2{\cdot}6990+{\cdot}9031-3{\cdot}5471\}$$
$$=\tfrac{1}{5}\{{\cdot}0550\}={\cdot}0110$$

$$\therefore\quad 1+\frac{x}{100}=\text{anti}\log {\cdot}0110=1{\cdot}026.$$
$$x=2{\cdot}6.$$

Hence the rate per cent per annum is *2 6*.

MISCELLANEOUS EXERCISES ON CHAPTER XI

1. Given that $\log 3={\cdot}47712$, how many digits are there in 3^{100}, and 30^{15}. (*P. U.*)

2. Prove that

$$\log\frac{16}{27}+\log\frac{27}{125}+\log\frac{125}{8}=\log 2.$$

3. Prove that

$$1.\log a+2.\log a^2+3\log a^3+4\log a^4+\ldots\ldots$$
$$\ldots\ldots+n\log a^n=\frac{n(n+1)(2n+1)}{6}\log a.$$

4. Solve the equations

(*i*) $2^x\,.\,3^{x+4}=7^x$.

(*ii*) $2^{2x+1}\,.\,3^{2x+2}=7^{4x}$.

(*iii*) $5^{3x+2}\,.\,7^{2x+1}=11^x$.

5. Find approximately the amount of a sum of Rs. 100 at 4% per annum after 50 years. What would the amount be if the interest were payable half yearly ?

6. Find the mean proportional between

$$\sqrt[3]{347{\cdot}3}\text{ and }\sqrt[5]{256{\cdot}4}.$$

7. If x, y, z be in H. P., prove that

$$\log(x+z)+\log(x-2y+z)=2\log(x-z).$$

8. A. G. P. and an H. P., have the same pth, qth and rth terms a, b, c. Prove that

$$a(b-c)\log a+b(c-a)\log b+c(a-b)\log c=0,$$

[Hint: Let x, y be the first and c. r. for the G. P.

$\therefore \log a=\log xy^{p-1}=\log x+(p-1)\log y$. Similarly $\log b$ and $\log c$ are found.

$$\therefore \text{ the R. H. S.}=\Sigma a(b-c)[\log x+\overline{p-1}\log y]$$
$$=\log x\,\Sigma a(b-c)+\log y\,\Sigma a(b-c)\,\overline{p-1}$$

1st term is identically zero and the second shall be zero by applying the conditions of H. P.]

ANSWERS TO EXERCISES IN CHAPTER XI

Exercise XI (A)

3. (a) 2, 0, 1. (b) $\bar{2}$, $\bar{1}$, 3, 4.
4. 4, $\frac{4}{3}$, $\frac{5}{4}$. 5. $\bar{4}$·6989, 3·5815.
6. 30, 32nd. 7. (i) $x=1·6$ nearly. (ii) $x=2·05$.
8. $x=2·915$, $y=·422$.
9. (i) 1·59. (ii) 3·32.

Exercise XI (B)

1. (i) 2·4713. (ii) 3·7378.
2. (i) 1·59 nearly. (ii) 2·09. (iii) $\bar{1}$·81.
3. (i) 658·2. (ii) ·03830. (iii) ·003271.
4. (i) ·3129. (ii) ·6183. 5. ·0484544.
6. (i) ·4143. (ii) ·1234. 7. ·0009342.

Exercise XI (C)

1. (i) $\bar{1}$·4287. (ii) $\bar{1}$·8289. (iii) 2·2182.
2. 36°52′ nearly, 21°48′.
3. (i) $\bar{1}$·8355, (ii) $\bar{1}$·5228, (iii) 1·4753.
4. (i) 9·7989, (ii) 10·1387.
5. ·4971 (nearly). 7. 43°19′34·71″.
8. (i) 1·5554, (ii) 2·9733.

Miscellaneous Exercises XI

1. (i) 48, (ii) 23.
4. (i) 28·5, (ii) 93, (iii) −·8162.
5. (i) Rs. 707·9 and Rs. 724·4. 6. 4·616.

CHAPTER XII

SOLUTION OF TRIANGLES

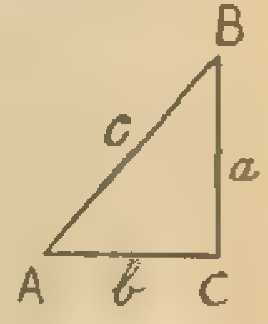
121. In every triangle there are six elements, namely, the three sides and three angles. The solution of triangles is the process by which when any three elements of a triangle are given, at least one of them being a side, the triangle is generally completely known. When the three angles are given, we cannot determine the lengths of the sides, but only the ratios of the sides. The reason for this is that the three angles are not three independent elements, as when two angles of a triangle are known the third is also known.

We shall begin with the solution of right-angled triangles and shall suppose C to be the right-angled triangle.

Case I. To solve a right-angled triangle, having given the hypotenuse and one acute angle.

Suppose the hypotenuse c and the acute angle A to be given. Then

$$B=90-A$$



Also $\frac{a}{c}=\sin A \quad \therefore \quad a=c \sin A.$

$\therefore \log a=\log c+\log \sin A$

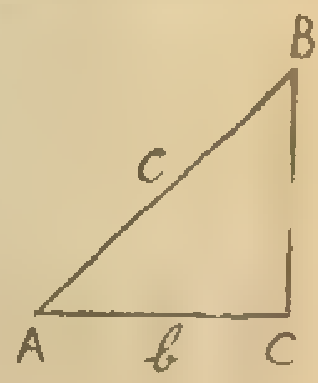
$b=c \sin B.$

$\therefore \log b=\log c+\log \sin B.$

Thus B, a, b are determined.

Case II. To solve the right-angled triangle having given the hypotenuse c and one side, say, b.

B is given by $\sin B=\frac{b}{c}$



$\therefore \log \sin B=\log b-\log c.$

$\therefore$ B is known

A is $90-B$, $\quad \therefore$ A is known.

$a=c \cos B$

$\therefore \log a=\log c+\log \cos B.$

Thus A, B, a are determined.

Case III. To solve right-angled triangle given a side and an acute angle.

Suppose a, A are given. Then

$$B=90-A.$$

$$\frac{a}{c}=\sin A \quad \therefore \quad c=\frac{a}{\sin A}$$

$$\therefore \quad \log c=\log a-\log \sin A$$

$$\tan \quad A=\frac{a}{b}$$

$$\therefore \quad b=\frac{a}{\tan A}$$

$$\therefore \quad \log b=\log a-\log \tan A$$

Thus B, c, b are determined.

Case IV. To solve a right-angled triangle having given the two sides.

Here a and b are given. Then

$$\tan A=\frac{a}{b}$$

$$\therefore \quad \log \tan A=\log a-\log b$$

$\therefore$ A is known

$$B=90^\circ-A$$

$$\frac{a}{c}=\sin A$$

$$\therefore \quad c=\frac{a}{\sin A}$$

$$\text{or} \quad \log c=\log a-\log \sin A$$

Thus A, B, c are determined.

Example 1. *Solve the right-angled triangle given that $c=32{\cdot}3$, $a=16{\cdot}7$.*

$$\sin A=\frac{a}{c}=\frac{16{\cdot}7}{32{\cdot}3}$$

$$\therefore \quad \log \sin A=\log 16{\cdot}7-\log 32{\cdot}3=1{\cdot}2227-1{\cdot}5092$$

$$=1{\cdot}7135=\log \sin 31^\circ\, 8'$$

$$\therefore \quad A=31^\circ 8'$$

$$B=90^\circ-A=90^\circ-31^\circ\, 8'=58^\circ\, 52'$$

$$\log b=\log c+\log \sin B=1{\cdot}5092+1{\cdot}9325$$

$$\therefore \quad =1{\cdot}4417=\log 27{\cdot}65$$

$$b=27{\cdot}65.$$

Example 2. *Solve the right-angled triangle given that* $b=75{\cdot}48$, $B=23°\ 9'$

$A=90°-B=90°-23°\ 9'=66°\ 51'$

$\tan B=\dfrac{b}{a}$

$\therefore \quad \log a=\log b-\log\tan B$

$=\log 75{\cdot}48-\log\tan 23°\ 9'$

$=1{\cdot}8779-\bar{1}{\cdot}6310=2{\cdot}2469$

$=\log 176{\cdot}6$

$\therefore \quad a=176{\cdot}6$

$\sin B=\dfrac{b}{c}$

$\therefore \quad \log c=\log b-\log\sin B$

$=\log 75{\cdot}48-\log\sin 23°\ 9'$

$=1{\cdot}8779-\bar{1}{\cdot}5946$

$=2{\cdot}9833=\log 192{\cdot}0$

$\therefore \quad c=192.$

EXERCISE XII (A)

Solve the following right-angled triangles, right-angled at C :—

1. $A=30°13'$ $\quad a=16{\cdot}83$
2. $a=321{\cdot}4$, $\quad b=123{\cdot}9$
3. $a=1{\cdot}732$, $\quad B=82°\ 13'$
4. $c=71{\cdot}45$, $\quad a=23{\cdot}75$
5. $c=29{\cdot}9$, $\quad A=33°22'$

12·2 Solution of Oblique-Angled Triangles.

We shall now discuss the case of oblique-angled triangles. The different cases to be considered are

(1) when the three sides are given ;
(2) two sides and the included angle are given ;
(3) one side and two angles are given ;
(4) two sides and the angle opposite to one of them are given.

Case 1. To solve a triangle having given the three sides.

Let the three sides be a, b, c. Then s, $s-a$, $s-b$, $s-c$ can be found, where s denotes half the sum of the sides.

Then as already proved

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

and similar formulae are true for the other half-angles. The formulae for the tangents of half the angles will be the best to use with logarithms, because then we only require the logarithms of s, $s-a$, $s-b$, $s-c$ in order to find all the angles, whereas if we use the formulae for sine and cosine, we shall require in addition the logarithms of the sides.

$$\text{Now } \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\therefore \log\tan\frac{A}{2}=\frac{1}{2}[\log(s-b)+\log(s-c)-\log s-\log(s-a)]$$

from which A is determined from tables. Similarly B is found from the formulae for $\tan\frac{B}{2}$. Then $C=180-A-B$.

Note. We can also use the formulae $\cos A=\frac{b^2+c^2-a^2}{2bc}$ etc., provided that the calculations are easy.

Example. *Solve the triangle whose sides are 32·65, 42·81 and 64·92.*

Let $a=32{\cdot}65$, $b=42{\cdot}81$, $c=64{\cdot}92$

$$2s=32{\cdot}65+42{\cdot}91+64{\cdot}92=140{\cdot}38$$

$$\therefore\ s=70{\cdot}19,\quad s-a=37{\cdot}54,\quad s-b=27{\cdot}38$$

$$s-c=5{\cdot}27$$

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\therefore \log\tan\frac{A}{2}=\tfrac{1}{2}[\log(s-b)+\log(s-c)-\log s-\log(s-a)]$$

$$=\tfrac{1}{2}[\log 27{\cdot}38+\log 5{\cdot}27-\log 70{\cdot}19-\log 37{\cdot}54]$$

$$=\tfrac{1}{2}[1\cdot4375+\cdot7218-1\cdot8463-1\cdot5745]$$
$$=\tfrac{1}{2}[\bar{2}\cdot7385]\bar{1}\cdot36925$$
$$\therefore\quad \frac{A}{2}=13^\circ\ 10'$$
$$\therefore\quad A=26^\circ\ 20'$$

Also $\log\tan\dfrac{B}{2}=\tfrac{1}{2}[\log(s-a)+\log(s-c)-\log s-\log(s-b)]$

$$=\tfrac{1}{2}[1\cdot5745+\cdot7218-1\cdot8463-1\cdot4375]$$
$$=\tfrac{1}{2}\times\bar{1}\cdot0125$$
$$=\tfrac{1}{2}[\bar{2}+1\cdot0125]$$
$$=\bar{1}\cdot50625$$
$$\therefore\quad \frac{B}{2}=17^\circ.\ 47' \qquad \therefore\ B=35^\circ\ 34'$$
$$C=180-(A+B)$$
$$=118^\circ 6'$$

EXERCISE XII (B)

1. Solve the triangle, having given $a=60$, $b=160$, $c=180$. (*P. U.*)

2. If $a=32$, $b=40$, $c=66$, find the angle C. (*P. U. 1911*)

3. If $a=5484$, $b=5140$, $c=3625$, find A. (*P. U. 1913*)

4. The sides of a triangle are 16, 20, 33. Find the greatest angle. (*P. U. 1939*)

[**Hint:** The greatest angle is opposite to the greatest side].

5. Find the smallest angle for the triangle whose sides are 18·1, 18·9, 18·5.

[**Hint:** The smallest angle is opposite to the smallest side].

6. Find the smallest angle of a triangle whose sides are proportional to 7 : 8 : 9.

[**Hint:** Let $a=7k$, $b=8k$, $c=9k$, $s=12k$.

then $\tan\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

$$=\sqrt{\frac{4k.3k}{12k.5k}}=\sqrt{\frac{4.3}{12.5}}$$

$=\sqrt{\dfrac{1}{5}}$, we find that 'k' automatically goes off in the result.

7. Using logarithmic tables, solve the triangle whose sides are 31·9, 56·31, 40·27. (*P. U. 1948*)

8. Given $a=31$, $b=42$, $c=57$, solve the triangle. (*P. U. 1947*)

Case II. To solve a triangle given two sides and the included angle.

Let the sides a, b and the included angle C be given.

Then $\tan \frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$

$$=\frac{a-b}{a+b}\tan\frac{A+B}{2}$$

$\therefore \log\tan\frac{A-B}{2}=\log(a-b)-\log(a+b)$

$+\log\tan\frac{A+B}{2}$

where $A+B=180-C$ is known.

From this relation we can find $\frac{A-B}{2}$. $A+B$ is already known and therefore we can find A and B.

To find the third side C, apply the law of sines.

$$\frac{c}{\sin C}=\frac{b}{\sin B}$$

$\therefore \log c=\log b+\log\sin C-\log\sin B$

From which side c is known.

Example. *Solve the triangle given* $a=21$, $b=11$, $C=34^\circ\ 42'\ 30''$.

We have $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$

$$=\frac{10}{32}\cot 17^\circ\ 21'\ 15''$$

$\therefore \quad \log\tan\frac{A-B}{2}=\log 10-\log 32+\log\tan 72^\circ\ 38'\ 45'$

$$=1-1{\cdot}5051+{\cdot}5051=0$$

$\therefore \quad \tan\frac{A-B}{2}=1=\tan 45^\circ$

$\therefore \quad \frac{A-B}{2}=45^\circ$ or $A-B=90^\circ$

Also $A+B=180°-34° 42' 30''=145° 17' 30''$

$\therefore \quad A=117° 38' 45'', B=27° 38' 45''$

Also $\log c=\log b+\log \sin C-\log \sin B$
$=\log 11+\log \sin 34° 42' 30''-\log \sin 27° 38' 45''$
$=1{\cdot}0414-\bar{1}{\cdot}7554-\bar{1}{\cdot}6666$
$=1{\cdot}1302=\log 13{\cdot}50$

$\therefore \quad c=13{\cdot}5.$

Note. The formula $c^2=a^2+b^2-2ab \cos C$ may be used to determine the third side. But the formula in its present form is not suited to logarithmic calculations; it is seldom used. It may, however, be used when the sides are represented by small numbers.

EXERCISE XII (C)

1. If $b=8$, $c=5$, $A=36° 52'$, solve the triangle. (*P. U. 1936*)

2. If $b=540$, $c=420$, $A=52° 6'$, find B and C. (*P. U. 1940*)

3. Given $a=3'$, $c=6'$, $\angle B=36° 20'$ solve the triangle. (*P. U. 1934*)

[Note. When the sides are given as $a=3'$, $c=6'$ it means $a=3$ ft. and $c=5$ ft.]

4. In a triangle $\angle A=94° 16'$, $b=5038$, $c=6840$. Find $\angle B$ and $\angle C$. (*P. U. 1938*)

5. In a triangle $a=456{\cdot}12$, $b=296{\cdot}26$, $c=74° 20'$, find A and B. (*P. U. 1950*)

6. In a triangle $b=130$, $c=72$, $\angle A=42°$, solve the triangle.

Case III. Given one side and two angles, to solve the triangle.

Let A, B and c be given. Then to find C, apply $C=180°-(A+B)$.

To find 'a', apply $a=\dfrac{c \sin A}{\sin C}$

$\therefore \log a=\log c+\log \sin A-\log \sin C$, which determines a.

Similarly to find b, apply the relation $b=\dfrac{c \sin B}{\sin C}$.

Example. *Given $a=10{\cdot}2$, $B=25°\ 21'$, $C=82°\ 12'$, solve the triangle.*

$$\angle A=180-(B+C)=180-107°\ 32'=72°\ 28'$$

Now $b=\dfrac{a \sin B}{\sin A}$

$\therefore$ $\log b=\log a+\log \sin B-\log \sin A$

$=\log 10{\cdot}2+\log \sin 25°\ 21'-\log \sin 72°\ 28'$

$=1{\cdot}0086+\bar{1}{\cdot}6313-\bar{1}{\cdot}9793$

$={\cdot}6606=\log 4{\cdot}577$

$\therefore$ $b=4{\cdot}577$

And $c=\dfrac{a \sin C}{\sin A}$

$\therefore$ $\log c=\log a+\log \sin C-\log \sin A$

$=\log 10{\cdot}2+\log \sin 82°\ 12'-\log \sin 72°\ 28'$

$=1{\cdot}0086+\bar{1}{\cdot}9960-\bar{1}{\cdot}9793$

$=1{\cdot}0253=\log 10{\cdot}60$

$\therefore$ $c=10{\cdot}60.$

EXERCISES XII (D)

1. If $B=88°\ 36'$, $C=31°\ 51'$ $a=53$, solve triangle. (*P. U. 1935*)

2. If $A=14°\ 42'$ $B=56°\ 17'$, $C=12{\cdot}27$, solve the triangle.

3. If $a=33{\cdot}57$, $A=42°\ 17'$, $C=71°\ 10'$, solve the triangle.

4. Given $A=70°$, $B=55°$, $a=150$, solve the triangle, (*P. U. 1947-S*)

5. Given $A=60°$, $C=53°$, $a=152$, solve the triangle. (*P. U. 1932*)

6. In a triangle $B=64°\ 23'$, $C=72°\ 43'$, $a=18{\cdot}92$, solve the triangle. (*P. U.*)

Case IV. Given two sides of a triangle and angle opposite to one of them, to solve the triangle.

Let a, b and A be given. Then to find B, we use the relation.

$$\sin B=\frac{b \sin A}{a} \qquad \ldots\ldots(1)$$

$\therefore$ $\log \sin B=\log b+\log \sin A-\log a$ $\qquad \ldots\ldots(2)$

$\therefore$ B is determined.

If $B = x^\circ$ satisfies this equation, then $B = 180^\circ - x^\circ$ will also satisfy this equation, and therefore there will be two values of B. Any of these values will be admissible if when added to the given angle A, the sum is $< 180^\circ$, for otherwise the angle C will become negative or zero.

Now $C = 180^\circ - (A + B)$ and to find c we use the relation

$$c = a \frac{\sin C}{\sin A}$$

$\therefore$ $\log c = \log \sin C + \log a - \log \sin A$, which will determine c.

Thus B, C and c are determined.

Note. In equation (1) $\dfrac{b \sin A}{a}$ may be $>$, $=$ or < 1, *i.e.* $\log b + \log \sin A - \log a$ may $>$, $=$ or < 0.

(*i*) If $\dfrac{b \sin A}{a} > 1$ there will be solution of the triangle as $\sin B > 1$.

(*ii*) If $\dfrac{b \sin A}{a} = 1$, $\sin B = 1$, and $B = 90^\circ$, and there is only one solution.

(*iii*) If $\dfrac{b \sin A}{a} < 1$, $\sin B < 1$, and there are two values of B. We have to see whether one or both the values are admissible.

(α) If $a > b$, $B < A$ and only the acute value of B is permissible and hence there is only one solution.

(β) If $a = b$, $A = B$ and only one acute value of B is permissible.

(γ) If $a < b$, $B > A$ and both the values of B are permissible. In this case A will be acute. This is called the **ambiguous case**.

The Geometrical discussion will be given at the end of this chapter.

Example 1. *Given $b = 6$, $c = 5$, $B = 65^\circ$, solve the triangle.*

Here the angle opposite to the greater side is given and therefore there is no ambiguity ; C is acute.

$$\log \sin C = \log c - \log \sin B - \log b$$
$$= \cdot 6990 - 1\cdot 9499 - \cdot 7782$$
$$= 1\cdot 8707$$
$$\therefore \quad C = 47^\circ\ 57'$$

Now $\quad A = 180^\circ - (B + C)$
$= 69^\circ\ 3'$

Also $\quad \log a = \log c + \log \sin A - \log \sin C$
$= \cdot 6990 + \bar{1}\cdot 9703 - \bar{1}\cdot 8707$
$= \cdot 7986$

$\therefore \quad a = 6\cdot 29.$

Example 2. *Given $b = 6\cdot 2$, $c = 8\cdot 5$, $B = 40^\circ$, solve the triangle.*

Here the angle opposite to the smaller side is given, therefore there may be two possible solutions.

We have $\log \sin C = \log c + \log \sin B - \log b$
$= \cdot 9294 + \bar{1}\ 8081 - \cdot 7924$
$= \bar{1}\cdot 9451$

$\therefore \quad C = 61^\circ\ 48'$ or $118^\circ\ 12'$.

Now $\quad A = 78^\circ\ 12'$ or $21^\circ\ 48'$

Also $\quad \log a = \log b - \log \sin B + \log \sin A$
$= \cdot 7924 - \bar{1}\cdot 8081 + (\bar{1}\ 9907$ or $\bar{1}\cdot 5698)$
$= \cdot 9750$ or $\cdot 5541$

$\therefore \quad a = 9\cdot 441$ or $3\cdot 582$

Thus the two solutions are

(*i*) $C = 61^\circ\ 48'$, $A = 78^\circ\ 12'$, $a = 9\cdot 441$

(*ii*) $C = 118^\circ\ 12'$, $A = 21^\circ\ 48'$, $a = 3\cdot 582$

Example 3. *Solve the triangle given $c = 8\cdot 4$, $a = 7\cdot 16$ $A = 68^\circ\ 48'$.*

$$\sin C = \frac{c \sin A}{a}$$

$\therefore \quad \log \sin C = \log c + \log \sin A - \log a$
$= \log 8\ 4 + \log \sin 61^\circ\ 48' - \log 7\cdot 16$
$= \cdot 9243 + \bar{1}\cdot 9696 - \cdot 8549$
$= \cdot 8939 - \cdot 8549 = \cdot 0390$

a positive quantity, showing that $\sin C$ is > 1 and C cannot be found. Hence no triangle is possible.

EXERCISE XII (E)

1. Solve the triangle having given.

(*i*) $a = 7$, $\quad b = 10$, $A = 51^\circ$ *(P. U. 1937)*
(*ii*) $a = 11$, $b = 17$, $A = 30^\circ\ 21'$ *(P. U. 1937)*

2. Solve the triangle

$a=42\cdot24$, $b=47\cdot75$, $A=21^\circ\ 6'$ (*P. U.* 1912)

3. Find the two solutions for the triangle in which

$a=175\cdot2$, $c=215\cdot4$, $A=27^\circ\ 19'$

4. Given $b=31\cdot27$, $c=20\cdot54$, $C=47^\circ\ 15'$, solve the triangle.

5. If $B=88^\circ\ 30'$, $C=31^\circ\ 54'$, $a=53$, solve the triangle. (*P. U.* 1935)

6. If $C=421\cdot9$, $a=531\cdot4$, $A=70^\circ\ 15'$, solve the triangle. (*P. U.* 1942-S)

123. The cases we have already examined may be called the four standard cases in solution of triangles. However a triangle may be fixed in other ways, as for example, by its base, its height and one of the base angles, or in general by three independent quantities connected with the triangle of which at least one must be a length. Some of these cases are considered below :—

Example 1. *Given the base, height and an angle at the base, to solve the triangle.*

Let h be the height, a the base and B the given angle. Then

$$h=c\sin B$$

$\therefore$ c is known.

Thus we have two sides a, c and the included angle B, which is the standard case II.

Example 2. *Given the perimeter and two angles, to solve the triangle.*

Let A and B be the given angles. Then

$$C=180-(A+B)$$

$$\text{Now } \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$=\frac{a+b+c}{\sin A+\sin B+\sin C},$$

$$\therefore \quad a=\frac{2s\sin A}{\sin A+\sin B+\sin C}$$

where $2s$=the perimeter.

Similarly we can find b and c.

Example 3. *Given the base, one of the base angles, and the sum of the other two sides to solve the triangle.*

Let a, B, $b+c$ be known. Then

$$b^2=c^2+a^2-2ac\cos B.$$

i.e., $(b-c)(b+c)=a^2-2ac\cos B$, which is another relation between b and c. Hence we find b and c. Now the three sides are known, the angles can be determined.

Example 4. *Given the base, the angle opposite to the base and the sum of the two other sides, to solve the angle.*

Let a, A, $b+c$ be given, then

$$\frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos\frac{A}{2}\cos\frac{B-C}{2}}$$

$$=\frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}}$$

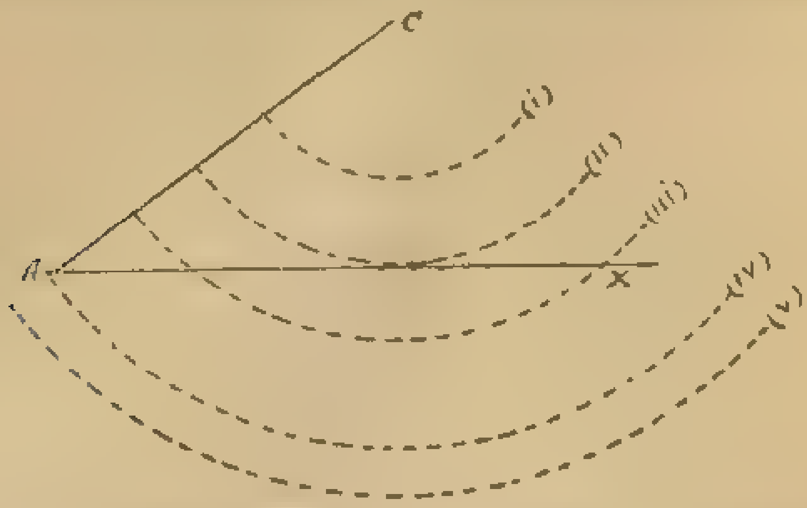
from which $\frac{B-C}{2}$ is known. As $\frac{B+C}{2}=90-\frac{A}{2}$ is also known, the angles B and C can be determined. The sides b and c can be determined by using the sine formula.

12·4. To construct the triangle when two sides and one of the angles opposite to one of them are given.

L. a, b, A be given.

Let AX be any line. Make angle CAX=the given angle and cut off AC=the given side b. With C as centre and radius equal to the given side a, describe an arc. This arc will cut the line AX in a point B. Then ABC will be the given triangle. Five different cases can arise which are given below :—

(*i*) The arc may not cut the line AX in which case it is



impossible to construct the triangle. This corresponds to the case in which $\frac{b \sin A}{a} > 1$.

(*ii*) The arc may just touch the line AX, in which case the angle B will clearly be a right angle. There is only one solution in this case. This corresponds to the case in which

$$\frac{b \sin A}{a} = 1.$$

(*iii*) The arc may cut the line AX in two points, both of which lie in AX. In this case there are two positions of B and consequently there are two triangles. This corresponds to the case in which $\frac{b \sin A}{a} < 1$ and $a < b$.

(*iv*) The arc may cut the line AX in two points in which one of the points of intersection coincides with A. In this case there is only one position of B and there is only one solution of the problem. This corresponds to the case in which $\frac{b \sin A}{a} < 1$ and $a = b$.

(*v*) The arc may cut the line AX in two points which lie on opposite sides of A. In this case there will be two positions for B but in one of the triangles the angle A will be the external angle. Hence there will be only one solution of the triangle. This corresponds to the case in which $\frac{b \sin A}{a} > 1$ and $a > b$.

MISCELLANEOUS EXERCISES ON CHAPTER XII

1. The sides of a triangle are 2, 3, 4; find the greatest angle, having given

$\log 2 = \cdot 3010$, $\log 3 = \cdot 4771$,

$\log \tan 52°14' = \cdot 1108$, $\log \tan 52°15' = \cdot 1111$. (*D. U. 1940*)

2. Solve the triangle, given that

$$\text{area} = 2437,\ a = 79,\ c = 97.$$

[**Hint:** $\frac{1}{2} ac \sin B = 2437$

$\therefore \quad \sin B = \dfrac{2 \times 2437}{79 \times 97}$. There are two values of B etc.]

3. Two sides of a triangle are in the ratio of 5 : 3 and the included angle is 120°. Find the other angle and the ratio of the third side to one of the given sides. (*D. U. 1937*)

[**Hint:** Apply Napier's analogy and suppose the sides are $5k$ and $3k$. $3k$ will cancel.]

4. The perimeter of a triangle is 9, $A = 46°34'$, $B = 104°28'$, solve the triangle.

[**Hint:** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = \dfrac{a+b+c}{\sin A + \sin B + \sin C}$, which is known.]

5. Two sides of a triangle are respectively 7 feet and 8 feet and the included angle contains 22°. Find the other two angles, having given $\log 15 = 1\cdot1761$, L $\cot 11° = 10\cdot7113$, L $\tan 18°\,55' = 9\cdot5349$ and difference for $1' = \cdot0004$.

6. If the angles of a triangle are in A. P., and the lengths of the greatest and the least side be 6 and 4 feet respectively, find the length of the third side and the angles, given that

$$\log 2 = \cdot30100,\ \log 3 = \cdot4771213$$

and L $\log \tan 19°6' = 9\cdot5394287$,

difference for the minute $= 4084$.

[**Hint:** The middle angle $= 60°$.]

7. Two sides of a triangle are $\sqrt{3}+1$ and $\sqrt{3}-1$ and the included angle is 60°. Find the other side and the angles.

[**Note:** When sides are given in irrational numbers (like $\sqrt{3}+1$ and $\sqrt{3}-1$), we should not take logs but calculate the result directly.]

8. If a, b, A be the given parts (of a triangle) and c_1 and c_2 two values of the third side, show that $c_1+c_2=2b\cos A$, $c_1c_2=b^2-a^2$. (*D. U.*)

9. In a triangle ABC, b, c, $\angle B$ are given, also $b<c$, show that $(a_1-a_2)^2+(a_1+a_2)^2\tan^2 B=4b^2$, where a_1, a_2 are the two values of the third side.

[Hint: $\cos B=\dfrac{c^2+a^2-b^2}{2ac}$

$\therefore\quad a^2-(2c\cos B)\,a+(c^2-b^2)=0$

$\therefore\quad a_1+a_2=$ sum of the roots $=2c\cos B$ and $a_1a_2=c^2-b^2$

$\therefore\quad (a_1-a_2)^2=(a_1+a_2)^2-4a_1a_2=4c^2\cos^2 B-4(c^2-b^2)$

Substitute in the question the values of a_1+a_2 and a_1-a_2].

ANSWERS TO EXERCISES IN CHAPTER XII

Exercise XII (A)

1. $B=57°47'$; $C=31°57$, $b=26{\cdot}71$.
2. $A=68°55'$, $B=21°5'$, $c=344{\cdot}4$.
3. $A=7°47'$, $b=12{\cdot}67$. $c=12{\cdot}78$.
4. $A=17°31'$, $B=72°29'$ $b=68{\cdot}14$.
5. $B=56°38'$, $a=16{\cdot}44$, $b=24{\cdot}98$.

Exercise XII (B)

1. $A=19°12'$, $B=61°12'$, $C=99°36'$.
2. $C=132°34'$. 3. $60°10'$. 4. $132°34'$. 5. $57°52'$.
6. $48°12'$.
7. $44°24'$ (nearly) $33°42'$ (nearly) $101°54'$ (nearly).
8. $101°36'$, $46°12'$, $32°12'$.

Exercise XII (C)

1. $B=106°16'$, $C=36°52'$, $a=5$.
2. $B=78°18'$, $C=49°36'$.
3. $A=26°23'$, $C=117°17'$. $b=3{\cdot}99$.
4. $B=34°51'$, $C=50°53'$. 5. $A=68°25'$, $B=37°15'$.
6. $B=105°48'$, $C=32°12'$, $a=90{\cdot}38$.

Exercise XII (D)

1. $A=59°30'$, $b=61{\cdot}51$, $c=32{\cdot}51$.
2. $a=32{\cdot}94$, $b=108$, $C=109°1'$.
3. $b=45{\cdot}77$, $c=47{\cdot}22$. $B=66°33'$.
4. $C=55°$, $b=c=131{\cdot}6$. 5. $b=123{\cdot}2$, $c=112{\cdot}8$, $C=47°$.
6. $b=25{\cdot}06$, $C=26°54$, $A=42°54'$.

Exercise XII (E)

1. (*i*) Impossible.
(*ii*) $B=51^\circ 20'$ or $128^\circ 40'$, $C=98^\circ 19'$ or $20^\circ 59'$, $c=21{\cdot}54$ or $c=7{\cdot}796$.
2. $B=24^\circ 1'$ or $155^\circ 59'$, $C=134^\circ 53'$ or $2^\circ 55'$ $C=83{\cdot}12$ or $5{\cdot}936$.
3. $B_1=34^\circ 20'$; $C_1=118^\circ 21'$; $c_1={\cdot}467$; $B_2=145^\circ 40'$. $C_2=70^\circ 2'$; $c_2={\cdot}336$.
4. Impossible. 5. $b=61{\cdot}51$, $c=32{\cdot}51$, $A=59^\circ 30'$.
6. $B=61^\circ 24'$, $C=48^\circ 21'$. $b=495{\cdot}8$.

Miscellaneous Exercises on Chapter XII

1. $104^\circ 28' 20''$:
2. $b=51{\cdot}83$ or $165{\cdot}8$, $A=54^\circ 21'$ or $17^\circ 29'$. $B=39^\circ 32'$ or $140^\circ 30'$.
3. $39^\circ 12' 40''$ or $20^\circ 47' 20''$; ratio $7{\cdot}745$ to ${\cdot}3$.
4. $C=28^\circ 10'$, $a=3$, $b=4$, $c=2$ omitting decimals.
5. $88^\circ 28'$, $69^\circ 32^\circ$. 6. $2\sqrt{7}$. $79^\circ 6' 24''$, 60°, $40^\circ 53' 36''$.
7. $B=105^\circ$, $C=15^\circ$ and $a=\sqrt{6}$.

CHAPTER XIII

HEIGHTS AND DISTANCES

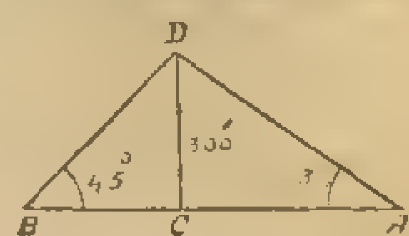
13. We have already seen some of the simple applications of elementary Trigonometry to the measurements of heights and distances. We are now in a position to take up a few more applications of Trigonometry and show in what way Trigonometry is useful in engineering, survey and map-making, etc. The following examples illustrate how a knowlege of the magnitude of certain angles, heights or distances helps us in determining the heights and distances of certain inaccessible objects.

Two types of problems are to be considered :—

(*i*) Those in which the objects lie in one plane.

(*ii*) Those in which the objects lie in different planes.

Example 1. *From a lighthouse the angles of depression of two ships on opposite sides of the lighthouse are observed to be 30° and 45°. If the height of the lighthouse be 300 feet, find the distance between the ships if the line joining them passes through the foot of the lighthouse.*



(*P. U. 1941*)

Let A, B be the ships and CD the lighthouse. Then $\angle CAD = 30°$ and $\angle CBD = 45°$. From the ΔACD,

$$AC = CD.\cot 30° = 300\sqrt{3} \text{ feet},$$

and from the ΔBCD,

$$BC = CD.\cot 45° = 300 \text{ feet},$$

$\therefore$ the distance between the ships $= AB$

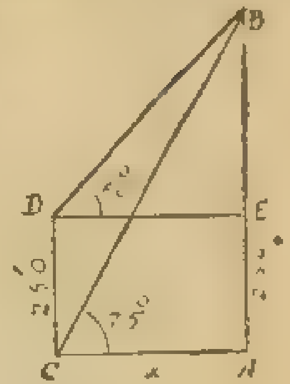
$$= AC + BC = 300\sqrt{3} + 300 \text{ feet}$$

$$= 300 \times 2{\cdot}732 = 819{\cdot}6 \text{ feet}.$$

Example 2. *The angles of the elevation of the top of a tower from the top and bottom of a building 250 feet high are 50° and 75°. Find the height of the tower.*

(*Delhi Uni. 1937*)

Let AB be the tower and CD the building. Let DE be $\perp$ AB. Then $\angle ACB$ and $\angle EDB$ are 75° and 50° respectively. If $AC = x$, then from the $\triangle ACB$, we have



$$x = h \cot 75^\circ,$$

where h is the height of the tower.

Also from the $\triangle BDE$,

$$x = DE = BE \cot 50 = (h - 250) \cot 50^\circ$$

$$\therefore\ (h-250)\cot 50^\circ = h \cot 75^\circ,$$

or $\quad h(\cot 50^\circ - \cot 75^\circ) = 250 \cot 50^\circ$

$$\therefore\ h = \frac{250 \cot 50^\circ}{\cot 50^\circ - \cot 75^\circ} = \frac{250 \cdot \cos 50^\circ \cdot \sin 75^\circ}{\sin 75^\circ \cos 50^\circ - \cos 75^\circ \sin 50^\circ}$$

$$= \frac{250 \cos 50^\circ \sin 75^\circ}{\sin(75^\circ - 50^\circ)} = \frac{250 \sin 40^\circ \sin 75^\circ}{\sin 25}$$

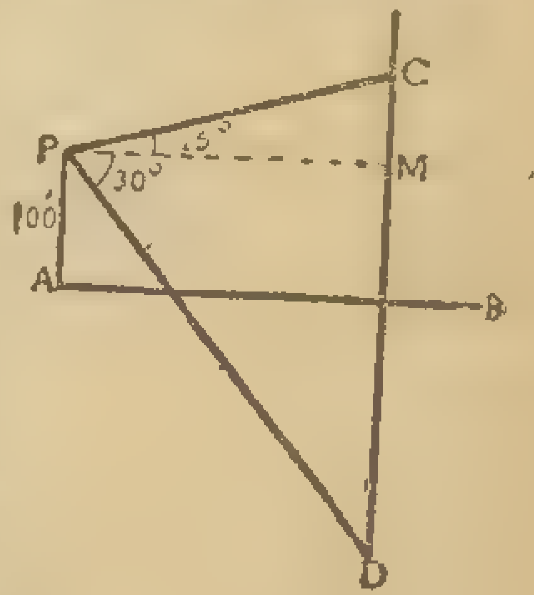
$$\therefore\ \log h = \log 250 + \log \sin 40^\circ + \log \sin 55^\circ - \log \sin 25^\circ$$

$$= 2{\cdot}3980 + \bar{1}{\cdot}8081 + \bar{1}{\cdot}9840 - \bar{1}{\cdot}6259$$

$$= 2{\cdot}5661 = \log 367{\cdot}4 \quad \text{Hence } h = 367{\cdot}4 \text{ feet.}$$

Example 3. *From a point 100 feet above the surface of a lake, the angular elevation of a peak is found to be 15°, and the angle of depression of the reflection of the peak is 30°. Find the height of the peak.* (*P. U. 1939*)

Let AB be the surface of the lake, P the point, C the peak and D its reflection. Let PM $\perp$ CD. Then $\angle CPM = 15^\circ$ and $\angle MPD = 30^\circ$. Let $PM = x$ and height of the peak $= h$. Then from the $\triangle PMC$,



$$x = (h - 100) \cot 15^\circ \qquad \ldots\ldots(1)$$

and from the $\triangle PMD$,

$$x = (h + 100) \cot 30^\circ \qquad \ldots\ldots(2)$$

From (1) and (2),

$$(h+100)\cot 30^\circ = (h-100)\cot 15^\circ,$$

$$\therefore\ \frac{h+100}{h-100} = \frac{\cot 15^\circ}{\cot 30^\circ}$$

$$\therefore\ \frac{h}{100} = \frac{\cot 15^\circ + \cot 30^\circ}{\cot 15^\circ - \cot 30^\circ} = \frac{\sin 45^\circ}{\sin 15^\circ}$$

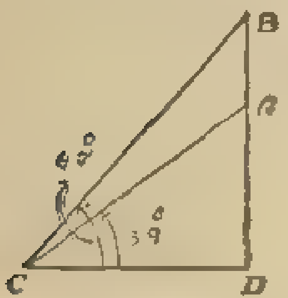
$$\therefore\ h = 100 \times \frac{\sin 45^\circ}{\sin 15^\circ}$$

$\therefore \quad \log a = \log 100 + \log \sin 45° - \log \sin 15°$

$= 2 + \bar{1}\cdot 8495 - \bar{1}\cdot 4030 = 2\cdot 4365$

$= \log 273\cdot 2 \qquad \therefore \quad h = 273\cdot 2$ feet.

Example 4. *The angles of elevations of 2 aeroplanes, one passing vertically over the other are seen by an observer to be 39° and 47° respectively. If the height of the lower aeroplane above the ground be 4049 feet, find the height of the upper aeroplane.* (*P. U. 1939*)



Let A, B be the planes and D the observer; then from the right-angled triangles ACD and BCD, we have

$$\frac{CD}{AD} = \cot 39°$$

and $$\frac{CD}{BD} = \cot 47°$$

Dividing we get

$$\frac{BD}{AD} = \frac{\cot 39°}{\cot 47°} = \frac{\tan 47°}{\tan 39°}$$

$\therefore \quad \log BD = \log AD + \log \tan 47° - \log \tan 39°$

$= \log 4049 + \log \tan 47° - \log \tan 39°$

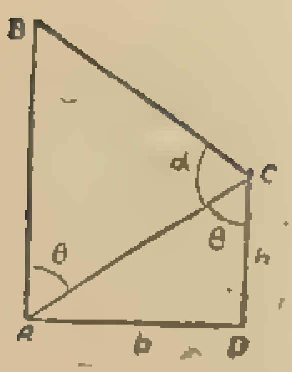
$= 3\cdot 6074 + \cdot 0303 - \bar{1}\cdot 9084$

$= 3\cdot 7293$

$= \log 5326$

$\therefore \quad BD = 5362$ feet

Example 5. *From a window on one side of a street a building on the other side is observed to subtend an angle α. If the width of the street be b feet and the height of the point of observation be h feet, show that the height of the building is* $\dfrac{(b^2+h^2)\sin\alpha}{b\cos\alpha + h\sin\alpha}$



Let AB be the building, C the window and AD the breadth of the street.

Then $\angle ACB = \alpha$, $AD = b'$, $CD = h'$

Let $\angle BAC = \angle ACD = \theta$; then from $\triangle ABC$

$$\frac{AB}{\sin\alpha} = \frac{AC}{\sin ABC} = \frac{AC}{\sin(180 - \theta - \alpha)}$$

$$= \frac{AC}{\sin(\theta + \alpha)} = \frac{AC}{\sin\theta\cos\alpha + \cos\theta\sin\alpha}$$

$$=\frac{AC}{\frac{b}{AC}\cos\alpha+\frac{h}{AC}\sin\alpha}=\frac{AC^2}{b\cos\alpha+h\sin\alpha}$$

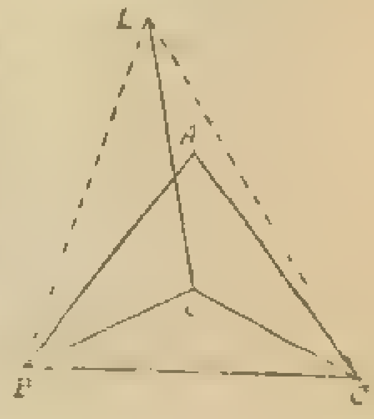
$$=\frac{b^2+h^2}{b\cos\alpha+h\sin\alpha}$$

$\therefore$ AB, the height of the building

$$=\frac{(b^2+h^2)\sin\alpha}{b\cos\alpha+h\sin\alpha}.$$

Example 6. *A pole, 100 feet high, stands in the centre of an equilateral triangle which is horizontal. From the top of the pole each side subtends an angle of 60°; prove that the length of the side of the triangle is $50\sqrt{6}$ feet.*

(*P. U. 1944*)



Let ABC be the equilateral $\triangle$ and O its centre. Let OL be the pole. Then $\angle BLC=60°$. Also BL = CL from symmetry (or from the congruent right-angled triangles OBL and OCL), therefore, $\angle LBC=\angle LCB=60°$ each. Hence $\triangle$ LBC is an equilateral triangle. If BC$=x$ feet; then LB$=x$ feet. Also OB$=\frac{2}{3}\times$ perp. of the $\triangle$

$$=\frac{2}{3}\times x.\sin 60$$

$$=\frac{x}{\sqrt{3}}$$

Now since OBL is a right-angled triangle,

$$OB^2+OL^2=BL^2$$

$$\therefore\quad \frac{x^2}{3}+100^2=x^2$$

or $\quad \frac{2x^2}{3}=100^2$, *i.e.*, $x=100\times\sqrt{\frac{3}{2}}$

$$=50\sqrt{6}\text{ feet.}$$

EXERCISE XIII

1. A ladder 30 feet long reaches to a distance 20 feet from the top of a flagstaff. At the foot of the ladder the elevation of the top is 60°. Find the height of the flagstaff.

(*P. U. 1940*)

2. A person walking along the straight bank of a river observes that an object on the other bank makes an angle

$22°48'$ with the bank. He walks a distance of 400 feet further and observes that the object now makes an angle of $69°15'$. Find the breadth of the river. (*P. U. 1936 Supp.*)

3. A tower 150 feet high stands on the top of a cliff 80 feet. At what point on the plane passing through the foot of the cliff must an observer place himself, so that the tower and the cliff may subtend equal angles, the height of the eye being 5 feet. (*P. U. 1928*)

[**Hint**: The bisector of the triangle divides the opposite side in the ratio of the other two sides.]

4. The angles of elevation of the top of the tower from the top and bottom of a building h feet high are α and β. Find a formula for the height of the tower suitable for logarithmic calculations.

Given $h=20$ feet, $\alpha=30°$, $\beta=31°$, calculate the height of the tower and its distance from the building.

5 The angles of elevation of the top of a tower from two points distant a and b from the base and in the same straight line with it are complementary. Prove that the height of the tower is $\sqrt{ab}$ and if α be the angle subtended at the top of the tower by the line joining these points; then

$$\sin \alpha=\frac{a-b}{a+b}. \qquad \text{(P. U. 1938)}$$

6. Standing on a seashore a man whose eyes are 24 feet above the water finds that the angle of elevation of a small cloud is $31°$ while the angle of depression of the cloud in the water is $32°$. Find the height of the cloud above the sea level. (*P. U. 1937*)

7. AB is the straight road leading to C, the foot of a tower, A being at a distance of 400 feet from C and B 250 feet nearer. If the angle of elevation of the tower at B be double of the angle of elevation at A; find the height of the tower and the angle of elevation at A. (*P. U. 1935*)

8 The angle of elevation of a tower from the point A due South of it is x and from a point B due East of A is y. If $AB=l$, show that the height h of the tower is given by $h^2(\cot^2 y-\cot^2 x)=l^2$. (*P. U. 1943*)

9. A vertical flagstaff stands on a horizontal plane, from a point distant 150 feet from its foot, the angle of elevation of its top is found to be 50°. Find the height of the flagstaff.

(P. U. 1942)

10. From a station B at the base of a mountain its summit A is seen at an elevation of 60°; after walking one mile towards the summit up a plane making an angle of 30° with the horizon of another station C, the angle BCA is observed to be 135°. Find the height of the mountain in yards.

11. The angular elevation of a tower at a place A due south of it is 30°; and at a place B, due west of A, and at a distance a from it the elevation is 18°. Show that the height of the tower is $\dfrac{a}{\sqrt{2+2\sqrt{5}}}$

12. A vertical tower PQ stands on a hill which is inclined to the vertical at an angle α. At two points A and B, a feet apart on the side of the hill, in the same vertical plane as the tower, the angles subtended by the tower are β and α. Show that the height of the tower is $\dfrac{a \sin \alpha \sin \beta}{\sin \alpha \sin (\alpha-\beta)}$.

13. A statue 30 feet high, standing on the top of column subtends at a point distant 150 feet in a horizontal line from the base of the column, the same angle at that subtended at the same point by a man 6 feet high standing at the base of the column. Find the height of the column. *(P. U. 1959-S)*

14. If the angle of elevation of a cloud from a point h feet above a lake be α, and the angle of depression of its reflection in the lake be β, prove that the height of the cloud above the lake is $\dfrac{h \sin (\alpha+\beta)}{\sin (\beta-\alpha)}$. *(D. U.)*

ANSWERS TO EXERCISE XIII

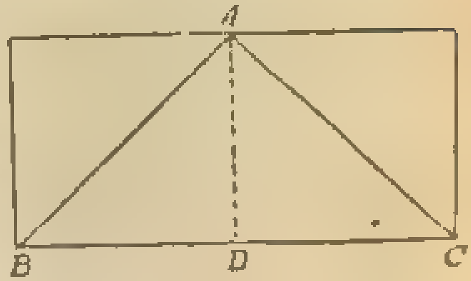
Exercise XIII

1. 31·55 ft. 2. 200 ft. 3. 141.
4. $\dfrac{h \cos \alpha \sin \beta}{\sin (\beta-\alpha)}$: 511 ft. nearly.
6. 1226 ft. 7. 200; $\tan^{-1}(\frac{1}{2})$.
9. 86·6 ft. 10. 4164 16 yards. 13. 298·7.

CHAPTER XIV

PROPERTIES OF TRIANGLES

14·1 Area of a given triangle.



Area of a triangle is half of the rectangle with the same base and altitude ; thus if ABC be any triangle and AD be the perpendicular from A on the opposite side, we have if $\triangle$ be the area,

$$\begin{aligned}\triangle &= \text{the area of the triangle}\\ &= \tfrac{1}{2}\text{BC}\cdot\text{AD},\\ &= \tfrac{1}{2}a\cdot\text{AD},\end{aligned}$$

and $\text{DA}=c \sin \text{B}$.

$\therefore \quad \triangle=\frac{1}{2}ac \sin \text{B}$. **(*P. U. 1947*)**

Thus area of a triangle is half the product of two sides $\times$ the sine of the included angle. Hence

$$\left.\begin{aligned}\triangle &= \tfrac{1}{2}bc \sin \text{A}\\ &= \tfrac{1}{2}ca \sin \text{B}\\ &= \tfrac{1}{2}ab \sin \text{C}\end{aligned}\right\} \quad \ldots\ldots(1)$$

Also we know that $\sin \text{A}=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$ ......(2)

$\therefore \quad \triangle=\sqrt{s(s-a)(s-b)(s-c)}$

[This is called Hero's Formula for the area of a triangle]

Since $\dfrac{a}{\sin \text{A}}=\dfrac{b}{\sin \text{B}}=\dfrac{c}{\sin \text{C}}$

$\therefore \quad b=\dfrac{a \sin \text{B}}{\sin \text{A}}, \quad c=\dfrac{a \sin \text{C}}{\sin \text{A}}$

Substituting these values in (1) we have

$$\triangle=\tfrac{1}{2}a^2.\frac{\sin \text{B} \sin \text{C}}{\sin \text{A}}=\tfrac{1}{2}a^2\frac{\sin \text{B} \sin \text{C}}{\sin (\text{B}+\text{C})} \quad \ldots\ldots(3)$$

(*P. U. 1936*)

Similarly

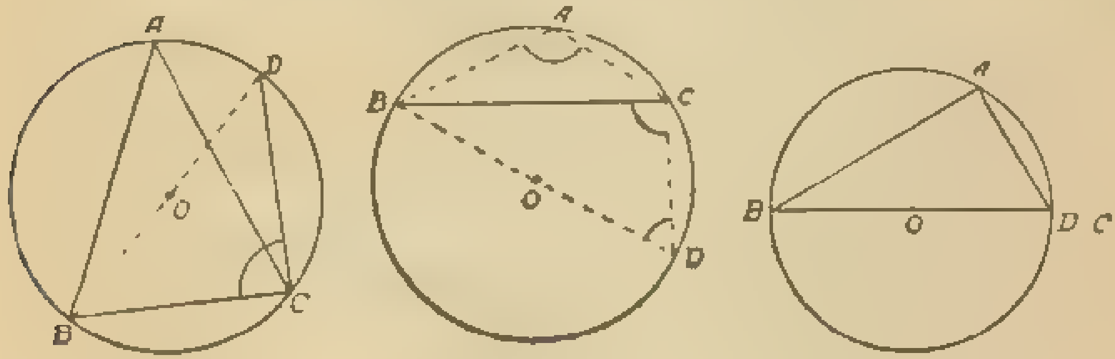
$$\Delta=\tfrac{1}{2}b^2.\frac{\sin C.\sin A}{\sin B}$$

$$=\tfrac{1}{2}c^2.\frac{\sin A.\sin B}{\sin C}$$

These formulae give the area in terms of one side and the angles.

14·2. To prove that in any triangle ABC,

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}. \qquad (P.\ U.\ 1947)$$



(*i*) (*ii*) (*iii*)

Let A be an acute angle as in fig. (*i*) or obtuse angle as in fig. (*ii*) or a rt. angle as in fig. (*iii*).

Let O be the circumcentre. Join BO and let it meet the circumference in D. Join DC. $\angle BCD=90°$, being an angle in a semi-circle.

In fig. (*i*) $\frac{BC}{BD}=\sin BDC=\sin A \quad \therefore \frac{a}{2R}=\sin A$

or $a=2R \sin A$

In fig. (*ii*) $\frac{BC}{BD}=\sin BDC=\sin (180-A)=\sin A$

$\therefore \qquad \frac{a}{2R}=\sin A$ or $a=2R\sin A$

In fig. (*iii*) $\frac{BC}{BD}=\frac{BC}{BC}=1=\sin 90°=\sin A$

$\therefore \quad a=2R\sin A$

$\therefore$ in each fig. $a=2R\sin A$

$$\therefore \quad R=\frac{a}{2\sin A}$$

Similarly

$$R=\frac{b}{2\sin B}=\frac{c}{2\sin C}$$

Cor. Since $a=2R\sin A$

$\therefore$ any chord of a circle $=$ (diameter) $\times$ sine of the angle the chord subtends at the circumference.

14·21. To prove that in any triangle

$$R=\frac{abc}{4\triangle}. \qquad (P.\ U.\ 1947)$$

$$R=\frac{a}{2\sin A} \text{ (from Art. 14·2)}$$

$$=\frac{a}{2\times\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}}$$

$$=\frac{abc}{4.\sqrt{s(s-a)(s-b)(s-c)}}=\frac{abc}{4\triangle}.$$

14·22. To prove that in any triangle

$$R^2=\frac{\triangle}{2\sin A.\sin B.\sin C}$$

$$\triangle=\tfrac{1}{2}bc\sin A$$
$$=\tfrac{1}{2}(2R\sin B).(2R\sin C).\sin A$$
$$=2R^2\sin A.\sin B.\sin C$$

$$\therefore\quad R^2=\frac{\triangle}{2\sin A.\sin B.\sin C}.$$

EXERCISE XIV (A)

Prove that in any triangle

1. $a\cos A+b\cos B+c\cos C=4R\sin A\sin B\sin C.$ *(P. U. 1939)*

[**Hint:** $a=2R\sin A$ etc.

$\therefore$ L.H.S. $=\Sigma 2R\sin A\cos A=R(\sin 2A+\sin 2B+\sin 2C)$

$=R.[4.\sin A.\sin B.\sin C]$

2. $s=4R\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}.$ *(P. U. 1933)*

[L.H.S. $=\frac{1}{2}(a+b+c)$

$=\frac{1}{2}[2R\sin A+2R\sin B+2R\sin C]$

$=R[\sin A+\sin B+\sin C]=$ etc.]

3. $\sin A + \sin B + \sin C = \frac{s}{R}$

[L.H.S. $= \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}$ $\qquad \because a = 2R \sin A$ etc.

$= \frac{a+b+c}{2R} = \frac{s}{R}$].

4. A line is drawn through the vertex of a triangle ABC to meet BC in D. Show that the ratio of the radii of the circumcircles of the triangles ABD, ACD is $\frac{c}{b}$ (*P. U. 1937-S*)

[Hint: R of $\triangle ABC = \frac{AB}{2 \sin ADB} = \frac{c}{2 \sin ADB}$

R′ of $\triangle ACD = \frac{CA}{2 \sin ACD} = \frac{b}{2 \sin (180 - ADB)}$

$= \frac{b}{2 \sin ADB}$ etc.]

5. If the sides of a triangle are 45, 55, 70; calculate the area of the circumcircle. (*P. U. 1947*)

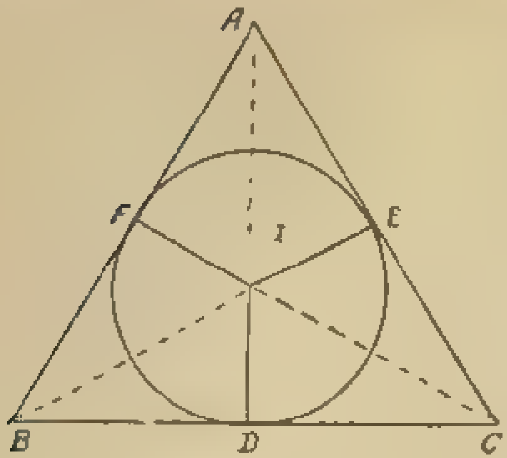
14·3. To prove that the radius of inscribed circle is given by the following formulae.

(*i*) $r = \frac{\Delta}{s}$. (*P. U.*)

(*ii*) $r = \dfrac{a \sin \frac{B}{2} . \sin \frac{C}{2}}{\cos \frac{A}{2}}$ = similar expressions.

(*iii*) $r = 4R \sin \frac{A}{2} . \sin^2 \frac{B}{2} . \sin \frac{C}{2}$. (*P. U*)

(*iv*) $r = (s-a) \tan \frac{A}{2}$ = similar expressions. (*P. U.*)



Proof. (*i*) Let BI and CI be the bisectors of the angles B and C meeting in I. Draw ID, IE, and IF perpendicular to the sides BC, CA, AB respectively. Then D, E, F are the points of contact, and ID = IE = IF $= r$.

Now Δ = Area of the $\triangle$ABC = Area of the $\triangle$BIC + Area of $\triangle$CIA + Area of $\triangle$AIB

$= \frac{1}{2}\text{BC}.\text{ID} + \frac{1}{2}\text{CA}.\text{IE} + \frac{1}{2}\text{AB}.\text{IF}$

$= \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$

$= \frac{1}{2}r(a+b+c) = sr \quad \therefore \quad r = \frac{\Delta}{s}$

$\therefore$ The radius of the inscribed circle of the $\triangle$ABC is equal to the area of the triangle divided by its semi-perimeter.

(*ii*) In terms of a side and the angles.

From the figure we have

$$\begin{aligned}
\text{BC} &= \text{BD} + \text{DC},\\
&= \text{DI} \cot \text{IBC} + \text{DI} \cot \text{ICB},\\
&= r \cot \frac{B}{2} + r \cot \frac{C}{2},\\
&= [r.] \left(\frac{\cos \frac{B}{2}}{\sin \frac{B}{2}} = \frac{\cos \frac{C}{2}}{\sin \frac{C}{2}} \right)\\
&= r.\frac{\sin \left(\frac{B}{2} + \frac{C}{2} \right)}{\sin \frac{B}{2} \sin \frac{C}{2}} = \frac{r \cos \frac{A}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}}
\end{aligned}$$

$$\therefore \quad r = \frac{a \sin \frac{B}{2}.\sin \frac{C}{2}}{\cos \frac{A}{2}}$$

$$\text{Similarly } r = \frac{b \sin \frac{A}{2} \sin \frac{C}{2}}{\cos \frac{B}{2}} = \frac{c \sin \frac{B}{2} \sin \frac{A}{2}}{\cos \frac{C}{2}}.$$

(*iii*) In terms of the angles and the radius of the circum-circle.

Since $a=2R\sin A$,

we have $$r=\frac{a\sin\frac{B}{2}.\sin\frac{C}{2}}{\cos\frac{A}{2}}=\frac{2R\sin A\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}$$

$$=4R\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}.$$

(*iv*) Since $\angle IBD=\frac{B}{2}$,

$$\tan\frac{B}{2}=\frac{ID}{BD}=\frac{r}{s-b}$$

[$\because$ $AE+BF+CD=AE+CE+BD=s$,

$\therefore$ $BD=s-b$, $CD=s-b$]

$\therefore$ $r=(s-b)\tan\frac{B}{2}$,

Similarly $r=(s-a)\tan\frac{A}{2}$,

$$=(s-c)\tan\frac{C}{2}.$$

Alternative Proof for (*ii*), (*iii*) and (*iv*).

(*ii*) $$r=\frac{a\sin\frac{B}{2}.\sin\frac{C}{2}}{\cos\frac{A}{2}}$$

$$\text{R.H.S.}=\frac{a\sqrt{\frac{(s-a)(s-c)}{ac}}.\sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$=\frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}}=\frac{\Delta}{s}=r.$$

(*iii*) $r=4R \sin \frac{A}{2} . \sin \frac{B}{2} . \sin \frac{C}{2}$

$$\text{R.H.S.}=4.\frac{abc}{4\triangle} . \sqrt{\frac{(s-b)(s-c)}{bc}} . \sqrt{\frac{(s-c)(s-a)}{ac}}$$

$$\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$=\frac{(s-a)(s-b)(s-c)}{\triangle}=\frac{\triangle^2}{\triangle s}=\frac{\triangle}{s}=r.$$

(*iv*) $r=(s-a) \tan \frac{A}{2}$

$$\text{R.H.S.}=(s-a)\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

$$=\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2}}=\frac{\triangle}{s}=\frac{\triangle}{s}=r.$$

EXERCISE XIV (B)

1. Given three sides $\begin{cases} a=58{\cdot}6 \\ b=64{\cdot}3 \\ c=52{\cdot}5 \end{cases}$

Calculate the area of the inscribed circle. (*P. U. 1945*)

[Hint : Area $=\pi(r)^2=\pi\left(\frac{\triangle}{s}\right)^2$

$$\therefore \quad A=\frac{\pi[\sqrt{s(s-a)(s-b)(s-c)}]^2}{s^2}=\frac{\pi(s-a)(s-b)(s-c)}{s}$$

Take logs etc.]

2. Prove that $Rr(\sin A+\sin B+\sin C)=\triangle$. (*P. U. 1940*)

[Hint : $\sin A=\frac{a}{2R}$ etc.]

3. Prove that $\triangle=4Rr\cos\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}$

[Hint : $\triangle=r.s=\frac{r}{2}(a+b+c)=\frac{r}{2}2R\sin A+\ldots+\ldots)$

$=Rr[\sin A+\sin B+\sin C]=$ etc.]

4. Prove that $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{1}{2Rr}$. (*P. U. 1937*)

[**Hint :** L.H.S. $=\dfrac{a+b+c}{abc}=\dfrac{2s}{abc}=\dfrac{2.\frac{\Delta}{r}}{4\Delta r}=\dfrac{1}{2Rr}$].

5. Prove that $\Delta=r^2.\cot\dfrac{A}{2}.\cot\dfrac{B}{2}.\cot\dfrac{C}{2}$ (*P. U.*)

[**Hint :** R.H.S. $=r^2.\sqrt{\dfrac{s(s-a)}{(s-b)(s-c)}}.\sqrt{\dfrac{s(s-b)}{(s-c)(s-a)}}.$

$$\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}=r^2\sqrt{\frac{s^3}{(s-a)(s-b)(s-c)}}$$

$$=\frac{r^2s^2}{\Delta}=\frac{\Delta^2}{\Delta}=\Delta].$$

6. Prove that $\cot\dfrac{A}{2}+\cot\dfrac{B}{2}+\cot\dfrac{C}{2}=\dfrac{s}{r}$.

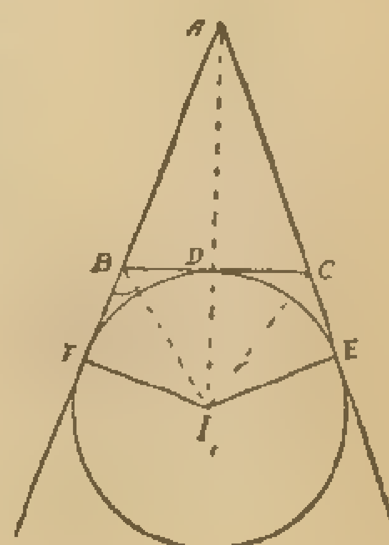
14·4. To prove that in any triangle, the described radius is given by the formulae :

(*i*) $r_1=\dfrac{\Delta}{s-a}$. (*P. U. 1941*)

(*ii*) $r_1=\dfrac{a\cos\frac{B}{2}.\cos\frac{C}{2}}{\cos\frac{A}{2}}$ (*P. U. 1942*)

(*iii*) $r_1=4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}.\cos\dfrac{C}{2}$.

(*iv*) $r_1=s\tan\dfrac{A}{2}=(s-b).\cot\dfrac{C}{2}$.



Proof. (*i*) Let I_1 be the *e*-centre opposite to A.

Join AI_1, BI_1, C_1, and draw I_1D, I_1E and I_1F perpendiculars to the sides of the triangle.

The $I_1D=I_1E=I_1F=r_1$

Now $\Delta=$area of ΔABC

$=\Delta I_1BA+\Delta I_1CA-\Delta I_1BC$

$=\frac{1}{2}AB.I_1F+\frac{1}{2}AC.I_1E-\frac{1}{2}BC.I_1D$

$=\frac{1}{2}cr_1+\frac{1}{2}br_1-\frac{1}{2}ar_1$

$=\frac{1}{2}r_1(b+c-a)$

$=\frac{1}{2}r_1(2s-2a)$
$=r_1(s-a)$

$\therefore \quad r_1=\dfrac{\Delta}{s-a}.$

(*iii*) Since $a=\text{BD}+\text{DC}=\text{I}_1\text{D}.\cot \text{I}_1\text{BD}+\text{I}_1\text{D}.\cot \text{I}_1\text{CD}$

$$=r_1\left(\cot \frac{180-\text{B}}{2}+\cot \frac{18-\text{C}}{2}\right)$$

$$=r_1\left(\tan \frac{\text{B}}{2}+\tan \frac{\text{C}}{2}\right)$$

$$=r_1\frac{\sin\left(\frac{\text{B}}{2}+\frac{\text{C}}{2}\right)}{\cos \frac{\text{B}}{2} \cos \frac{\text{C}}{2}}=\frac{r_1 \cos \frac{\text{A}}{2}}{\cos \frac{\text{B}}{2} \cos \frac{\text{C}}{2}}$$

$$\therefore \quad r_1=\frac{a \cos \frac{\text{B}}{2}.\cos \frac{\text{C}}{2}}{\cos \frac{\text{A}}{2}}$$

Similarly $r_2=\dfrac{b \cos \frac{\text{C}}{2} \cos \frac{\text{A}}{2}}{\cos \frac{\text{B}}{2}}$

and $r_3=\dfrac{c \cos \frac{\text{A}}{2}.\cos \frac{\text{B}}{2}}{\cos \frac{\text{C}}{2}}$

(*iii*) In terms of the angles and the radius of the circum-circle.

Since $a=2\text{R} \sin \text{A}$.

$$r_1=\frac{a \cos \frac{\text{B}}{2} \cos \frac{\text{C}}{2}}{\cos \frac{\text{A}}{2}}=\frac{2\text{R} \sin \text{A} \cos \frac{\text{B}}{2} \cos \frac{\text{A}}{2}}{\cos \frac{\text{A}}{2}}$$

$$\therefore \quad r_1=4\text{R} \sin \frac{\text{A}}{2} \cos \frac{\text{B}}{2} \cos \frac{\text{C}}{2}$$

Similarly $r_2=4R\sin\dfrac{B}{2}\cos\dfrac{C}{2}\cos\dfrac{A}{2}$

$r_3=4R\sin\dfrac{C}{2}\cos\dfrac{A}{2}\cos\dfrac{B}{2}$

(*iv*) Since $AE+AF=AC+CE+AB+BF$
$=AC+CD+AB+BD=AC+BB+BC=2s$

$\therefore\quad AE=AF=s$

$\therefore\quad \dfrac{I_1F}{AF}=\tan\ I_1AF=\tan\dfrac{A}{2}$

$\therefore\quad I_1F=AF\tan\dfrac{A}{2}$

$\therefore\quad r_1=s\ \tan\dfrac{A}{2}$

Similarly $r_2=s\tan\dfrac{B}{2}$ and $r_3=s\ \tan\dfrac{C}{2}$

(*v*) Since $AE=AF=s$,

$\therefore\quad BD=s-c$ and $CD=s-b$;

$\therefore\quad \dfrac{I_1D}{BD}=\tan\ I_1BD=\tan\dfrac{180-B}{2}=\cot\dfrac{B}{2}$

$\therefore\quad I_1D=BD.\cot\dfrac{B}{2}$

i.e., $r_1=(s-c)\cot\dfrac{B}{2}$

Similarly $r_1=(s-b)\cot\dfrac{C}{2}$

Hence $r_1=(s-b)\cot\dfrac{C}{2}=(s-c)\cot\dfrac{B}{2}$,

$r_2=(s-a)\cot\dfrac{C}{2}=(s-c)\cot\dfrac{A}{2}$,

$r_3=(s-a)\cot\dfrac{B}{2}=(s-b)\cot\dfrac{A}{2}$.

Alternative proof for (*ii*), (*iii*), (*iv*)

(*ii*) $\quad r_1=\dfrac{a\cos\dfrac{B}{2}.\cos\dfrac{C}{2}}{\cos\dfrac{A}{2}}$.

$$\text{R.H.S.}=\frac{a\sqrt{\dfrac{s(s-b)}{ca}}\cdot\sqrt{\dfrac{s(s-c)}{ab}}}{\sqrt{\dfrac{s(s-a)}{bc}}}$$

$$=\sqrt{\frac{s(s-b)(s-c)}{s-a}}=\sqrt{\frac{s(s-a)(s-b)(s-c)}{(s-a)^2}}$$

$$=\frac{\Delta}{s-a}=r_1.$$

(*iii*) $r_1=4R\sin\dfrac{A}{2}.\cos\dfrac{B}{2}.\cos\dfrac{C}{2}$

$$\text{R.H.S.}=4.\frac{abc}{4\Delta}.\sqrt{\frac{(s-b)(s-c)}{bc}}.\sqrt{\frac{s(s-b)}{ac}}.\sqrt{\frac{s(s-c)}{ab}}$$

$$=\frac{s(s-b)(s-c)}{\Delta}=\frac{s(s-a)(s-b)(s-c)}{(s-a)\,\Delta}$$

$$=\frac{\Delta^2}{(s-a)\Delta}=\frac{\Delta}{s-a}=r_1$$

(*iv*) (*a*) $\text{R.H.S.}=s.\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$

$$=\sqrt{\frac{s(s-b)(s-c)}{s-a}}=\sqrt{\frac{s(s-a)(s-b)(s-c)}{(s-a)^2}}$$

$$=\frac{\Delta}{s-a}=r_1$$

(*v*) (*b*) $\text{R.H.S.}=(s-b)\cot\dfrac{C}{2}=(s-b)\sqrt{\dfrac{s(s-c)}{(s-a)(s-b)}}$

$$=\sqrt{\frac{s(s-b)(s-c)}{(s-a)}}=\sqrt{\frac{s(s-a)(s-b)(s-c)}{(s-a)^2}}$$

$$=\frac{\Delta}{s-a}=r_1.$$

SOLVED EXAMPLES

1. Prove that $r_1+r_2+r_3-r=4R$. (*P. U. 1949*)

$$r_1+r_2+r_3-r=\frac{\Delta}{s-a}+\frac{\Delta}{s-b}+\frac{\Delta}{s-c}-\frac{\Delta}{s}$$

$$=\left(\frac{\Delta}{s-a}+\frac{\Delta}{s-b}\right)+\left(\frac{\Delta}{s-c}-\frac{\Delta}{s}\right)$$

$$= \Delta \left[\frac{2s-(a+b)}{(s-a)(s-b)} + \frac{c}{s(s-c)} \right]$$

$$= \Delta \left[\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)} \right]$$

$$[\because \quad 2s = a+b+c]$$

$$= \Delta c \left[\frac{s(s-c)+(s-a)(s-b)}{s(s-a)(s-b)(s-c)} \right]$$

$$= \frac{\Delta c}{\Delta^2} \left[2s^2 - s(a+b+c) + ab \right]$$

$$= \frac{\Delta c}{\Delta^2} \left[2s^2 - 2s^2 + ab \right]$$

$$= \frac{\Delta abc}{\Delta^2} = \frac{abc}{\Delta} = 4.\frac{abc}{4\Delta} = 4.R.$$

2. Prove that

(i) $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$ *(P. U. 1935)*

(ii) $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2} = \frac{a^2+b^2+c^2}{\Delta^2}$

(P. U. 1941, 48)

Proof (*i*) $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$

$$= \frac{3s-(a+b+c)}{\Delta} = \frac{s}{\Delta} = \frac{1}{r}.$$

(*iii*) $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2}$

$$= \frac{(s-a)^2}{\Delta^2} + \frac{(s-b)^2}{\Delta^2} + \frac{(s-c)^2}{\Delta^2} + \frac{s^2}{\Delta^2}$$

$$= \frac{(s-a)^2+(s-b)^2+(s-c)^2+s^2}{\Delta^2}$$

$$= \frac{4s^2-2s(a+b+c)+a^2+b^2+c^2}{\Delta^2}$$

$$= \frac{4s^2-2s.2s+a^2+b^2+c^2}{\Delta^2} = \frac{a^2+b^2+c^2}{\Delta^2}$$

3. Prove that

(i) $rr_1r_2r_3=\Delta^2$

$$\frac{\Delta}{s}\cdot\frac{\Delta}{s-a}\cdot\frac{\Delta}{s-b}\cdot\frac{\Delta}{s-c}=\frac{\Delta^4}{s(s-a)(s-b)(s-c)}=\frac{\Delta^4}{\Delta^2}=\Delta^2.$$

(ii) $r_1r_2+r_2r_3+r_3r_1=s^2.$ *(P. U. 1944)*

$$\frac{\Delta}{(s-a)}\cdot\frac{\Delta}{(s-b)}+\frac{\Delta}{(s-b)}\cdot\frac{\Delta}{(s-c)}+\frac{\Delta}{(s-c)}\cdot\frac{\Delta}{(s-a)}$$

$$=\frac{\Delta^2[(s-a)+(s-b)+(s-c)]}{(s-a)(s-b)(s-c)}$$

$$=\frac{\Delta^2(3s-a+b+c)}{(s-a)(s-b)(s-c)}=\frac{\Delta^2.s}{(s-a)(s-b)(s-c)}$$

$$=\frac{\Delta^2s^2}{s(s-a)(s-b)(s-c)}=\frac{\Delta^2s^2}{\Delta^2}=s^2.$$

(iii) $rr_1\cot\frac{A}{2}=\Delta$

$$rr_1\cot\frac{A}{2}=\frac{\Delta}{s}\cdot\frac{\Delta}{s-a}\cdot\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$=\frac{\Delta^2}{\sqrt{s(s-a)(s-b)(s-c)}}=\frac{\Delta^2}{\Delta}=\Delta$$

(iv) $\frac{r_1-r}{a}+\frac{r_2-r}{b}=\frac{c}{r_3}$ *(P. U. 1936-S)*

$$\frac{r_1-r}{a}=\frac{1}{a}\left(\frac{\Delta}{s-a}-\frac{\Delta}{s}\right)=\frac{\Delta}{s(s-a)}$$

Similarly $\frac{r_2-r}{b}=\frac{\Delta}{s(s-b)}$

$\therefore$ $$\text{L.H.S.}=\frac{\Delta}{s(s-a)}+\frac{\Delta}{s(s-b)}$$

$$=\frac{\Delta[(s-b)+(s-a)]}{s(s-a)(s-b)}$$

$$=\frac{\Delta(2s-b-a)(s-c)}{s(s-a)(s-b)(s-c)}$$

$$=\frac{\Delta.c.(s-c)}{\Delta^2}=\frac{c(s-c)}{\Delta}=\frac{c}{r_3}$$

4. The sides of a triangle are 34, 20, 42; find the radii of its escribed circles. (*P.U. 1950*)

$$2s=a+b+c=96 \qquad \therefore \quad s=48$$

$$\therefore \quad \Delta=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{48.14.28.6}$$

$$=\sqrt{16.3.14.14.2.2.3}=4\ 3.14.2=336$$

$$\therefore \quad r_1=\frac{\Delta}{s-a}=\frac{336}{14}=24$$

$$r_2=\frac{\Delta}{s-b}=\frac{336}{28}=12$$

$$r_3=\frac{\Delta}{s-c}=\frac{336}{6}=56$$

EXERCISE XIV (C)

1. The sides of a triangle are 16, 20 and 33 ft. Find the radius of the escribed circle corresponding to the greatest angle. (*P. U. 1939*)

2. Prove that in any triangle

(*i*) $\Delta=a\dfrac{rr_1}{r-r_1}$.

(*ii*) $s^2=\dfrac{r_1.r_2.r_3}{r}$.

(*iii*) $\Delta=r_2r_3\tan\dfrac{A}{2}$ (*P.U.*)

3. Prove that in any triangle

(*i*) $(r_1+r_2)\tan\dfrac{C}{2}=(r_3-r)\cot\dfrac{C}{2}=c$. (*P.U. 1944*)

(*ii*) $\dfrac{r.r_1}{r_2r_3}=\tan^2\dfrac{A}{2}$.

(*iii*) $\dfrac{r_2+r_3}{r_1-r}=\cot^2\dfrac{A}{2}$.

(*iv*) $\dfrac{r^3}{r_1r_2r_3}=\tan^2\dfrac{A}{2}.\tan^2\dfrac{B}{2}.\tan^2\dfrac{C}{2}$.

4. Prove that (*i*) $16R^2rr_1r_2r_3=a^2b^2c^2$.

(*ii*) $(r_1-r)(r_2-r)(r_3-r)=4Rr^2$.

5. If A, A_1, A_2, A_3 be the areas of the in-circle and ex-circles, show that $\dfrac{1}{\sqrt{A}}=\dfrac{1}{\sqrt{A_1}}+\dfrac{1}{\sqrt{A_2}}+\dfrac{1}{\sqrt{A_3}}$

(*D.U. 1934*)

[**Hint :** $A=\pi r^2$, $A_1=\pi r_1^2$ etc. Hence we want to prove that $\dfrac{1}{\sqrt{\pi}r}=\dfrac{1}{\sqrt{\pi}}\left[\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}\right]$ or $\dfrac{1}{r}=\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}$, which is already solved (example 2 Art. 14·4)].

MISCELLANEOUS SOLVED EXAMPLES

1. Prove that in any triangle

(*i*) $\cos A+\cos B+\cos C=1-\dfrac{r}{R}$.

(*ii*) $a\cot A+b\cot B+c\cot C=2(R+r)$.

(*D.U. 1942*)

(*i*) L.H.S. (from identities $=1+4\sin\dfrac{A}{2}.\sin\dfrac{B}{2}.\sin\dfrac{C}{2}$

$$=1+\frac{r}{R}\left[\because\ r=4R\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}\right]$$

(*ii*) L.H.S. $=\Sigma\dfrac{a}{\sin A}\cos A$

$$=2R(\cos A+\cos B+\cos C)$$

$$=2R\left(1+\frac{r}{R}\right)=2R(R+r).$$

2. Prove that $\dfrac{1}{c\sin B}+\dfrac{1}{a\sin C}+\dfrac{1}{b\sin A}=\dfrac{1}{r}$

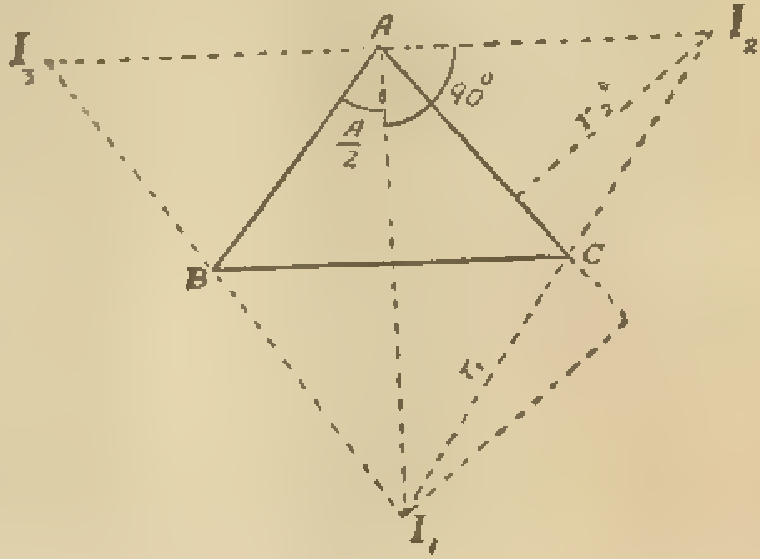
$$\text{L.H.S.}=\frac{1}{c.\dfrac{2\Delta}{ac}}+\frac{1}{a.\dfrac{2\Delta}{ab}}+\frac{1}{b.\dfrac{2\Delta}{bc}}$$

$$=\frac{a}{2\Delta}+\frac{b}{2\Delta}+\frac{c}{2\Delta}$$

$$=\frac{a+b+c}{2\Delta}=\frac{2s}{2\Delta}=\frac{s}{\Delta}=\frac{1}{r}.$$

3. Prove that the triangle formed by the e-centres

$$= 8R^2 . \cos\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2} \qquad (P.U.\ 1944\text{-}S)$$



$AI_2 = r_2 \sec\frac{A}{2}$,

$AI_3 = r_3 \sec\frac{A}{2}$

$AI_1 = r_1 \operatorname{cosec}\frac{A}{2}$.

$\therefore$ Area $= \frac{1}{2}(r_2 + r_3)$

$\sec\frac{A}{2} . r_1 \operatorname{cosec}\frac{A}{2}$

$$= \frac{r_1(r_2 + r_3)}{\sin A}$$

$$= \frac{4R \sin\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}\left[4R \cos\frac{A}{2} . \sin\frac{B}{2} . \cos\frac{C}{2} + 4R \cos\frac{A}{2} \cos\frac{B}{2} . \sin\frac{C}{2}\right]}{\sin A}$$

$$= 8R^2\, 2 \sin\frac{A}{2} . \cos\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}\left[\sin\left(\frac{B}{2} + \frac{C}{2}\right)\right] \sin A$$

$$= 8R^2 \sin A \cos\frac{B}{2} . \cos\frac{C}{2} . \cos\frac{A}{2} \Big/ \sin A$$

$$= 8R^2 . \cos\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}.$$

MISCELLANEOUS EXERCISES ON CHAPTER XIV

1. Show that $\Delta = \dfrac{a^2 - b^2}{2} . \dfrac{\sin A \sin B}{\sin(A - B)}$

$$= \frac{2abc}{a+b+c} . \cos\frac{A}{2} . \cos\frac{B}{2} \cos\frac{C}{2}.$$

2. Prove that $s = 4R \cos\frac{A}{2} . \cos\frac{B}{2} . \cos\frac{C}{2}$ (*P.U. 1933*)

3. Show that in any triangle the area of the in-circle is to the area of the triangle as $\pi : \cot\frac{A}{2} . \cot\frac{B}{2} . \cot\frac{C}{2}$.

4. If the escribed circle corresponding to A be equal to the circumcircle, prove that cos B+cos C=cos A.

(*P.U.* 1931)

$$\left[\textbf{Hint}: R=r_1=4R \sin\frac{A}{2}.\cos\frac{B}{2}.\cos\frac{C}{2}\right]$$

5. If 'r' be the in-radius of ΔABC, s the semi-perimeter and α, β, γ the distances of the vertices from the in-centre, prove that

(1) $r=\alpha\sin\frac{B}{2}.\sin\frac{C}{2}.\sec\frac{A}{2}$.

(2) $\alpha.\beta.\gamma.s=abc.r.$ (*P.U.* 1932)

$$\left[\textbf{Hint}: \text{From the figure, } \frac{IF}{AI}=\sin\frac{A}{2}.\right.$$

$$\left.\therefore\quad \alpha=\frac{r}{\sin\frac{A}{2}}\right].$$

6. O is the ortho-centre of the triangle ABC. Prove that the circum-radii of the triangles BOC, COA, AOB and ABC are equal.

[**Hint**: $R_1=\frac{a}{2\sin BOC}$ and $\angle BOC$=vertically opposite $\angle=180^\circ-A$.]

7. Prove that $R=\frac{abc(\cot A+\cot B+\cot C)}{a^2+b^2+c^2}$.

(*D.U.* 1935)

$$[\text{R.H.S.}=\frac{abc\left(\frac{\cos A}{\sin A}+\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}\right)}{a^2+b^2+c^2}$$

$$=\frac{abc\left(\frac{\frac{b^2+c^2-a^2}{2bc}}{\frac{a}{2R}}+\ldots+\ldots\right)}{a^2+b^2+c^2}=R.]$$

8. The sides of a triangle are 13, 12, 5 feet respectively. Calculate the lengths of R, r, r_1, r_2, r_3.

9. If I is the centre of the escribed circle opposite to angle A of the triangle ABC, prove that

$$AI_1 = 4R\cos\frac{B}{2}\cdot\cos\frac{C}{2}. \qquad (P.\ U.\ 1943\text{-}S)$$

Hint: $AI_1 = r_1 \operatorname{cosec}\frac{A}{2}$

$$=4R\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}\cdot\operatorname{cosec}\frac{A}{2}$$

$$=4R\cos\frac{B}{2}\cdot\cos\frac{C}{2}.]$$

10. Given the radii of the ex-circles, find the angles of a triangle. (*P. U. 1930*)

Hint: $r_1=\dfrac{\Delta}{s-a}$ $\therefore$ $s-a=\dfrac{\Delta}{r_1}$, $s-b=\dfrac{\Delta}{r_2}$, $s-c=\dfrac{\Delta}{r_3}$.

Adding $s=\Delta\left(\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}\right)$

$\tan\dfrac{A}{2}\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$ Δs cancel.]

11. Show that in an equilateral triangle

$$r : R : r_1 = 1 : 2 : 3.$$

12. If AL is the altitude of a triangle ABC, prove that

$$AL=\frac{BC}{\cot B+\cot C} \qquad (P.\ U.\ 1937)$$

13. If p_1, p_2, p_3 be the perpendiculars from the angular points of a triangle to the oppposite sides, show that

$$\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}.$$

14. Show that in a triangle

(*i*) $a=(r_2+r_3)\sqrt{\dfrac{r\cdot r_1}{r_2\cdot r_3}}$

(*ii*) $\sin\dfrac{A}{2}=\dfrac{r}{(r_2-r)(r_3-r)}$

(*iii*) $\tan\dfrac{A}{2}+\tan\dfrac{B}{2}+\tan\dfrac{C}{2}=\dfrac{r_1+r_2+r_3}{\sqrt{r_1r_2+r_2r_3+r_3r_1}}$

15. Show that in a triangle

(i) $4R = \sqrt{(b+c)^2 \sec^2 \frac{A}{2} + (b-c)^2 . \operatorname{cosec}^2 \frac{A}{2}}$.

(ii) $\frac{r}{R} = 4\left(\frac{s}{a} - 1\right)\left(\frac{s}{b} - 1\right)\left(\frac{s}{c} - 1\right)$

ANSWERS TO QUESTIONS IN CHAPTER XIV

Exercise XIV (B)

1. 858·6 Square Units.

Exercise XIV (C)

1. 78·6 ft. nearly.

Miscellaneous Exercises on Chapter XIV

8. $R = 6·5$; $r = 2$, $r_1 = 3$, $r_2 = 10$, $r_3 = 15$.

10. $\tan \frac{A}{2} = \frac{r_1}{\sqrt{\Sigma r_2 r_3}}$, $\tan \frac{B}{2} = \frac{r_2}{\sqrt{\Sigma r_2 r_3}}$, $\tan \frac{C}{2} = \frac{r_3}{\sqrt{\Sigma r_2 r_3}}$

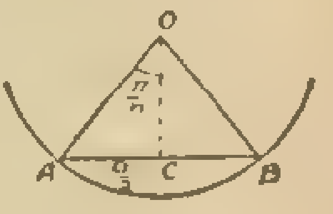
CHAPTER XV

REGULAR POLYGONS AND AREAS OF A CIRCLE

15·1. Def. A regular polygon is a polygon, the sides of which are equal to one another and so are the angles.

Let a regular polygon have n sides. Then each exterior angle $= \frac{4}{n}$. rt. $\angle$s. [$\because$ the sum of all exterior angles $=4$ rt. $\angle$s.] and each interior angle $= \frac{2n-4}{n}$ rt $\angle$s. [$\because$ the sum of all interior angles $=(2n-4)$ rt. $\angle$s.].

15·2. To find the radii of (i) the circumscribed and (ii) the inscribed circle of a regular polygon of n sides, each side being equal to a.



(i) Let AB be the side of a regular polygon of n sides and O the centre of the circumscribed circle. Join OA and OB.

Then OA $=$ OB $=$ Radius of the circumscribed circle $=$ R (say).

Draw OC $\perp$ AB $\therefore$ AC $=$ CB $= \frac{a}{2}$.

Now $\angle$ AOB $= \frac{1}{n}$ (4 rt. $\angle$s.) $= \frac{2\pi}{n}$.

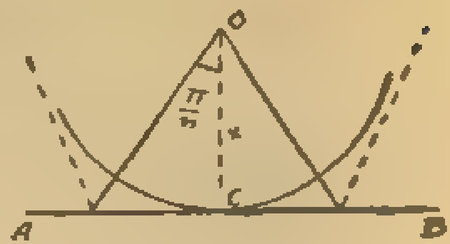
$\therefore \quad \angle \text{AOC} = \frac{1}{2}\angle \text{AOB} = \frac{1}{2} \cdot \frac{2\pi}{n} = \frac{\pi}{n}$.

$\therefore \quad \frac{\text{OA}}{\text{OC}} = \text{cosec AOC} = \text{cosec } \frac{\pi}{n}$.

$\therefore \quad \frac{\text{R}}{\frac{a}{2}} = \text{cosec } \frac{\pi}{n}$.

$\therefore \quad \text{R} = \frac{a}{2} \cdot \text{cosec } \frac{\pi}{2}$.

(*ii*) Let O be the centre of the inscribed polygon · OC ⊥ from O on AB = radius of the inscribed polygon (say r)



$$\angle AOC = \tfrac{1}{2}\angle AOB = \tfrac{1}{2}\frac{2\pi}{n} = \frac{\pi}{n}.$$

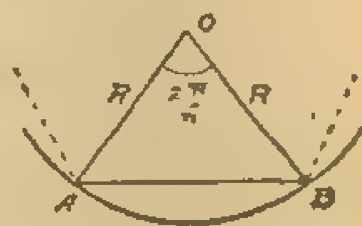
$$\therefore \quad \frac{OC}{AC} = \cot AOC = \cot\frac{\pi}{n}.$$

$$\therefore \quad r = \frac{a}{2}\cot\frac{\pi}{2}.$$

15·3. To find the area of a regular polygon of n sides

(*i*) in terms of the circum-radius (R)
(*ii*) in terms of the in radius (r)
(*iii*) in terms of the side 'a'.

(*i*) Let AB be the side of the polygon, and O the circumcentre.



Join OA, OB.
OA = OB = R.

Now area of the polygon $= n.\ \triangle AOB$

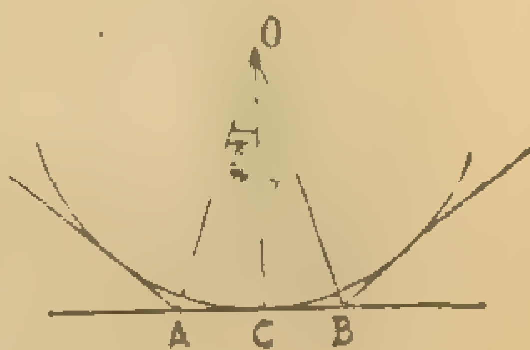
$= n.\tfrac{1}{2}OA.OB.\sin AOB$

[$\because \quad \triangle = \tfrac{1}{2}bc \sin A$]

$$= n \times \tfrac{1}{2}R.R.\sin\frac{2\pi}{n}$$

$$= \frac{nR^2}{2}\sin.\frac{2\pi}{n}.$$

(*ii*) Let AB be a side of a regular polygon of n sides, and O the centre of inscribed polygon.



Draw OC ⊥ AB.

$\therefore$ OC $= r$ and the angle AOB is bisected by OC $\therefore \angle AOC = \dfrac{\pi}{n}$.

Area of the polygon

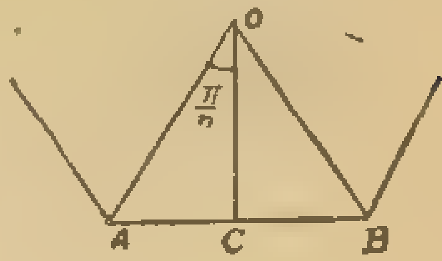
$= n.\triangle AOB$

$= n.\tfrac{1}{2}AB.OC$ [$\triangle = \tfrac{1}{2}$ base × alt.]

$= n \times AC.OC$

$$= n \times \left(r \tan\frac{\pi}{n}\right).r \left[\because \quad \frac{AC}{OC} = \tan\frac{\pi}{n}\right] = n.r^2.\tan\frac{n}{n}.$$

(*iii*) Let AB be the side of a polygon.



Draw OC ⊥ to AB.

$\therefore$ $AC = CB = \frac{1}{2}a$.

Area of the polygon $= n \times \triangle OAB$.

$= n \times \frac{1}{2} AB \times OC$.

$$= n \times \frac{a}{2} \times \frac{a}{2} . \cot \frac{\pi}{n}.$$

$$\left[\because \frac{OC}{AC} = \cot \frac{\pi}{n} \right]$$

$$= \frac{na^2}{4} . \cot \frac{\pi}{n}.$$

SOLVED EXAMPLES

1. One side of a regular decagon is 4 inches;

(*a*) find the radii of the circumscribed and inscribed circles;

(*b*) find the area of the polygon. (*P. U. 1937*)

Sol. (*a*) Refer to figures of Art. 15·2.

$$\frac{R}{\frac{a}{2}} = \operatorname{cosec} \frac{\pi}{n} \quad \therefore \quad R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}$$

$$\therefore \quad R = \frac{4}{2} . \operatorname{cosec} \frac{\pi}{10} = \frac{4}{2} . \operatorname{cosec} 18^\circ.$$

$$= \frac{4}{2} \times \frac{4}{\sqrt{5} - 1} = 8 \times \frac{(\sqrt{5}+1)}{(\sqrt{5}-1)(\sqrt{5}+1)} = 8 \times \frac{\sqrt{5}+1}{4}$$

$$= 2(\sqrt{5}+1)$$

Now $r = \frac{a}{2} \cot \frac{\pi}{n} = \frac{4}{2} . \cot \frac{\pi}{n} = 2 \cot 18^\circ$

$= 2 . \tan 72^\circ = 2 . (3{\cdot}0777)$ from tables $= 6{\cdot}1554$

(*b*) Area of the polygon from figure (*iii*) of Art. 15·3.

$= n \times \triangle OAB$

$= n \times AC \times OC$

$$= n \times \frac{a}{2} \times \frac{a}{2} . \cot \frac{\pi}{n} = \frac{10 \times 4^2}{4} . \cot \frac{\pi}{10}.$$

$= 40 . \cot 18^\circ$

$= 40 \tan 72^\circ = 40 . [3{\cdot}0777]$ from tables

$= 123.108.$

2. The sides of a triangle are respectively a side of a regular pentagon, hexagon and decagon inscribed in a circle; prove that the triangle is right-angled. (*P. U. 1938*)

Sol. Refer to fig. (*i*) of Art. 15·2.

$$R=\frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$$

or $$a=2R\sin\frac{\pi}{2}.$$

∴ The sides of the angle are

$$2R\sin\frac{\pi}{5},\ 2R\sin\frac{\pi}{6},\ 2R\sin\frac{\pi}{10}.$$

Now $$\left(2R\sin\frac{\pi}{6}\right)^2+\left(2R\sin\frac{\pi}{10}\right)^2$$

$$=(2R)^2\left[\sin^2\frac{\pi}{6}+\sin^2\frac{\pi}{10}\right]$$

$$=(2R)^2\left[\tfrac{1}{4}+\left(\frac{\sqrt{5}-1}{4}\right)^2\right]$$

$$=(2R)^2\left[\tfrac{1}{4}+\tfrac{1}{16}(5+1-2\sqrt{5})\right]$$

$$=(2R)^2\left[\tfrac{1}{4}+\tfrac{6}{16}-\frac{\sqrt{5}}{8}\right]$$

$$=(2R)^2\left[\frac{10-2\sqrt{5}}{16}\right]$$

$$=(2R)^2\sin^2\frac{\pi}{5}$$

$$=\left(2R\sin\frac{\pi}{5}\right)^2$$

∴ The △ is rt. ∠d. [∵ $a^2=b^2+c^2$].

EXERCISES XV (A)

1. The side of a regular polygon of twelve sides is 2 ft., find the radius of the circumscribed circle.

2. The area of a regular polygon of n sides inscribed in a circle is to that of the same number of sides circumscribing the same circle as 3 : 4. Find the value of n. (*P.U. 1941-S*)

[**Hint.** Area of the inscribed polygon

$$=n\times\tfrac{1}{2}R^2.\sin\frac{2\pi}{n},=A_1\ \text{(say)}$$

Area of the circumscribed polygon

$$=n\times\tfrac{1}{2}R^2.\sec^2\frac{\pi}{n}.\sin\frac{2\pi}{n}=A, \text{ (say)},$$

$$\therefore \quad \frac{A_1}{A_2}=\frac{1}{\sec^2\frac{\pi}{n}}=\tfrac{3}{4} \text{ or } \cos^2\frac{\pi}{n}=\tfrac{3}{4}.]$$

3. If an equilateral triangle and regular hexagon have the same perimeter, prove that their areas are as 2 : 3.

(*P. U. 1940-S*)

Hint. Area of a regular polygon in terms of its sides

$$=\frac{na^2}{4}.\cot\frac{\pi}{n}.$$

Let a and b be the side of the $\triangle$ and hexagon

$\therefore \quad 3a=6b$ or $a=2b$

$$\therefore \quad \frac{\text{Area of a triangle}}{\text{Area of hexagon}}=\frac{\frac{3}{4}.a^2.\cot\frac{\pi}{3}}{\frac{6}{4}.b^2.\cot\frac{\pi}{6}}.]$$

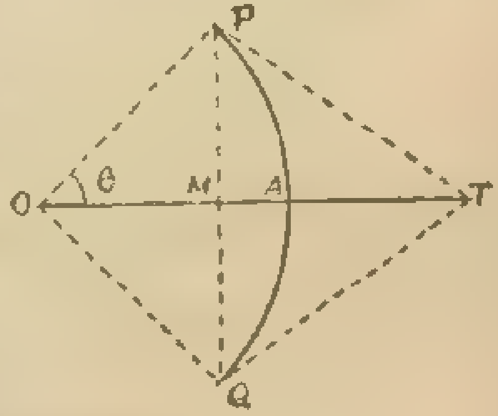
4. If a regular pentagon and a regular decagon have the same perimeter, prove that their areas are as $2:\sqrt{5}$.

5. If regular octogons be described about and in a given circle, show that the ratio of their areas are $3-2\sqrt{2}:1$.

(*P.U. 1932*)

Relation between the sine and the tangent of an angle with its Circular Measure.

15·4. If θ be the number of radians in an acute angle, to prove that $\sin\theta<\theta<\tan\theta$.



Let $\angle AOP=\theta$ radians, AP the arc of the circle whose centre is O and OP, its radius. Draw PM $\perp$ OA and let tangent at P meet OA produced in T.

Produce PM to meet the circle in Q, join OQ and QT.

Now PM=MQ [$\perp$ from the centre bisects the chord]

arc PA=arc QA [$\because \angle AOP=\angle QOA$]

PT=QT from similar $\triangle$s OPT and OQT.

Now chord PQ < arc PAQ < PT + TQ

$\therefore$ $2PM < 2 \text{ arc } AP < 2PT$

$\therefore$ $PM < \text{arc } AP < PT$

or $\dfrac{PM}{OP} < \dfrac{\text{arc } AP}{OP} < \dfrac{PT}{OP}$ [by dividing by OP]

$\therefore$ $\sin\theta < \theta < \tan\theta.$

15·41. If θ be the number of radians in an angle, to prove that $\underset{\theta \to 0}{\text{Lt}} \dfrac{\sin\theta}{\theta} = 1.$ (*P.U. 1942-49*)

$$\sin\theta < \theta < \tan\theta \text{ when } \theta =< \frac{\pi}{2} \qquad \text{[Art. 15.4]}$$

Dividing by $\sin\theta$

$$1 < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}.$$

or
$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

i.e. $\dfrac{\theta}{\sin\theta}$ lies between 1 and $\dfrac{1}{\cos\theta}$.

But $\underset{\theta \to 0}{\text{Lt}} \dfrac{1}{\cos\theta} = \dfrac{1}{1} = 1.$

$\therefore$ $\dfrac{\theta}{\sin\theta}$ lies between 1 and 1 when $\theta \to 0$.

Hence $\underset{\theta \to 0}{\text{Lt}} \dfrac{\theta}{\sin\theta} = 1$ or $\underset{\theta \to 0}{\text{Lt}} \dfrac{\sin\theta}{\theta} = 1.$

Cor. 1. If θ, the number of radians in an angle be very small, prove that Lt $\sin\theta = \theta$.

Lt $\dfrac{\sin\theta}{\theta} = 1$ when $\theta \to 0$.

$\therefore$ Lt $\sin\theta = \theta$, when θ is very small.

Cor. 2. Prove that $\underset{\theta \to 3}{\text{Lt}} \dfrac{\tan\theta}{\theta} = 1$ when θ is measured in radians and is very small.

$$\underset{\theta \to 0}{\text{Lt}} \frac{\tan\theta}{\theta} = \underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\cos\theta \cdot \theta}.$$

$$= \underset{\theta \to 0}{\text{Lt}} \frac{\sin\theta}{\theta} \cdot \underset{\theta \to 0}{\text{Lt}} \frac{1}{\cos\theta} = 1 \cdot \frac{1}{1} = 1.$$

Cor. 3. $\underset{n\to\infty}{\mathrm{Lt}} \dfrac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}$

Let $\dfrac{\theta}{n}=\phi \quad \therefore$ when $n\to\infty$, $\phi\to 0$

$$\therefore \quad \underset{n\to\infty}{\mathrm{Let}} \frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}} = \underset{\phi\to 0}{\mathrm{Lt}} \frac{\sin\phi}{\phi}=1.$$

Cor. 4. $\underset{n\to\infty}{\mathrm{Lt}}\, n\sin\dfrac{\theta}{n}=\theta$, θ measured in radians.

$$\mathrm{Lt}\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}=1 \quad \therefore \quad \underset{n\to\infty}{\mathrm{Lt}}\, n\frac{\sin\frac{\theta}{n}}{\theta}=1$$

$$\text{or} \quad \underset{n\to\infty}{\mathrm{Lt}}\, n\sin\frac{\theta}{n}=\theta.$$

SOLVED EXAMPLES

1. Find the limits of

$$(a)\ \frac{\sin x^\circ}{x} \text{ and } (b)\ \frac{\tan x^\circ}{x} \text{ when } x\to 0$$

Now $180^\circ=\pi$ radians

$$\therefore \quad x^\circ=\frac{\pi}{180}.x \text{ radians.}$$

$$(a) \therefore \quad \underset{x\to 0}{\mathrm{Lt}}\frac{\sin x^\circ}{x} = \underset{x\to 0}{\mathrm{Lt}}\frac{\sin\frac{\pi x}{180}}{x}$$

$$=\underset{x\to 0}{\mathrm{Lt}}\frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}\cdot\frac{\pi}{180}$$

$$=1\cdot\frac{\pi}{180}\left[\because \text{ If } x\to 0, \frac{\pi}{180}.x\to 0\right]=\frac{\pi}{180}$$

(b) $$\underset{x\to 0}{\text{Lt}}\frac{\tan x^\circ}{x}=\underset{x\to 0}{\text{Lt}}\frac{\tan\frac{\pi x}{180}}{x}$$

$$=\underset{x\to 0}{\text{Lt}}\frac{\tan\frac{\pi x}{180}}{\frac{\pi x}{180}}\cdot\frac{\pi}{180}$$

$$=\underset{\frac{\pi x}{180}\to 0}{\text{Lt}}\frac{\tan\frac{\pi x}{180}}{\frac{\pi x}{180}}$$

$$=1\cdot\frac{\pi}{180}=\frac{\pi}{180}.$$

2. A tower subtends an angle of 15′ at the eye of the observer 4 miles distant. Find approximately its height. $\left(\pi=\frac{22}{7}\right)$.

Let MP be the tower and O the observer's eye.

$\therefore$ $\angle MOP=15'$.

Now $\frac{PM}{OM}=\tan POM=\tan 15'$

$\therefore$ $PM=OM\tan 15'$

$$=OM.\tan\left(\frac{15}{60}\times\frac{\pi}{180}\right)$$

$$=OM.\frac{15}{60}\times\frac{\pi}{180}\left[\underset{\theta\to 0}{\text{Lt}}\frac{\tan\theta}{\theta}=1\text{ or }\underset{\theta\to 0}{\text{Lt}}\tan\theta=\theta\right]$$

$$=4\times 1760\times 3\times\frac{15}{60}\times\frac{1}{180}\times\frac{22}{7}$$

$=92.2$ ft. nearly.

3. Euler's theorem. Prove that

$$\sin\theta=\theta\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3}\ldots\ldots\infty \quad (P.U.\ 1940)$$

$$\sin\theta=2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}$$

$$=2.2\sin\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2}$$

$$=2^2.\sin\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^2},\cos\frac{\theta}{2}$$

$$=2^3.\sin\frac{\theta}{2^3}\cdot\cos\frac{\theta}{2^3}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2}$$

....

....

$$=2^n.\sin\frac{\theta}{2^n}.\cos\frac{\theta}{2^n}.\cos\frac{\theta}{2^{n-1}}\ldots\ldots\cos\frac{\theta}{2^2}.\cos\frac{\theta}{2}$$

$$\therefore\quad \sin\theta=2^n.\sin\frac{\theta}{2^n}.\cos\frac{\theta}{2}.\cos\frac{\theta}{2^2}\ldots\ldots\cos\frac{\theta}{2^n}$$

Let $n\to\infty$

$$\therefore\quad \underset{n\to\infty}{\text{Lt}}\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}=1 \quad\text{or}\quad \underset{n\to\infty}{\text{Lt}}\frac{2^n\sin\frac{\theta}{2^n}}{\theta}=1$$

$$\text{or}\quad \underset{n\to\infty}{\text{Lt}}\ 2^n\sin\frac{\theta}{2^n}=\theta$$

$$\therefore\quad \sin\theta=\theta.\cos\frac{\theta}{2}.\cos\frac{\theta}{2^2}\ldots\ldots\infty.$$

EXERCISE XV (B)

1. Prove that (*i*) $\underset{x\to 0}{\text{Lt}}\dfrac{\sin x'}{x}=\dfrac{\pi}{10800}$

(*ii*) $\underset{x\to 0}{\text{Lt}}\dfrac{\sin x''}{x}=\dfrac{\pi}{64800}$.

2. Find approximately the value of

(*i*) $\sin 10''$ to 6 places of decimal ;

(*ii*) the value of $\sin 1^\circ$ to 5 places of decimal. (*P. U.*)

[Hint. (*i*) $\sin 10'' = \sin\left(\frac{10}{60\times 60}\times\frac{\pi}{180}\right)$

$$= \sin \frac{\pi}{64800} = \frac{\pi}{64800} \text{ nearly} = \frac{3\cdot141592}{64800}$$

$= \cdot 000048$].

3. Prove that $\cos\theta$ lies between $1 - \frac{\theta^2}{2}$ and 1

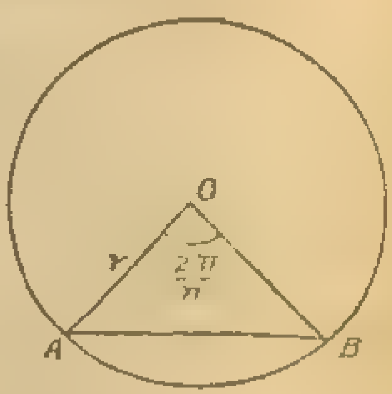
4. Prove that $\sin\left(\frac{\pi}{4}+\theta\right) = \frac{1+\theta}{\sqrt{2}}$ approximately, when θ is very small.

5. A tower 44 feet high subtends an angle of $35'$ at a point A on the ground ; find the distance of A from the tower.

6. A church tower at a distance of 2 miles subtends an angle of $1^\circ 5' 6''$. Find, approximately, its height.

AREA OF A CIRCLE

15·5. To prove that the area of a circle is πr^2 :



Let O be the centre of a circle. Inscribe a regular polygon of n sides in the circle, and let AB be a side of the polygon. Join OA and OB.

Then the area of the polygon

$= n \times \triangle OAB$

$= n \times \frac{1}{2}\, r.r \sin\frac{2\pi}{n}$ [$\triangle = \frac{1}{2}bc\sin A$]

$= n \times \frac{r^2}{2} . \sin\frac{2\pi}{n}$.

Let the number of sides of the polygon be increased indefinitely, so that the area of the polygon becomes the area of the circle.

∴ Area of the circle

$$= \underset{n\to\infty}{\mathrm{Lt}} \frac{nr^2}{2} . \sin\frac{2\pi}{n}$$

$$= \underset{n\to\infty}{\text{Lt}} r^2 . \frac{\sin\frac{2\pi}{n}}{\frac{2}{n}}$$

$$= \underset{n\to\infty}{\text{Lt}} r^2 \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}.\pi = r^2.1.\pi \quad \left[\because \underset{\theta\to\infty}{\text{Lt}} \frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}} = 1 \right]$$

$$= xr^2.$$

Cor. Show that the circumference of a circle is 2πR.

The perimeter of the polygon

$$= n.a = n.\text{AB}.$$

$$= n.2.\text{R} \sin\frac{\pi}{n}.$$

$\therefore$ circumference of the circle

$$= \underset{n\to\infty}{\text{Lt}} .2n\text{R}.\sin\frac{\pi}{n}$$

$$= \underset{n\to\infty}{\text{Lt}} 2\text{R}.\frac{\sin\frac{\pi}{n}}{\frac{1}{n}}$$

$$= 2\text{R} \underset{n\to\infty}{\text{Lt}} \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}.\pi$$

$$= 2\text{R}.1.\pi$$

$$= 2\pi\text{R}.$$

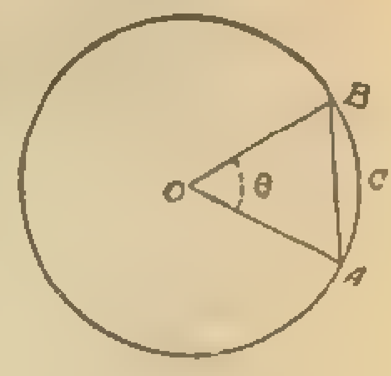
156. To find the area of the sector of a circle.

Let AOB be a sector, where angle AOB $=\theta$ radians.

Now $\dfrac{\text{area of the sector}}{\text{area of the circle}} = \dfrac{\angle \text{AOB}}{4 \text{ rt } \angle\text{s}} = \dfrac{\theta}{2\pi}$

$\therefore$ area of the sector $= \dfrac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2.\theta.$

Cor. To find the area of a segment of a circle.



Let ABC be the segment of a circle.

Then the area of the segment $\overline{ACB}$ = area of the sector AOB − area of the triangle AOB.

Let r be the radius and the angle AOB = θ radians.

$\therefore$ Area of the segment $\overline{ACB}$

$$= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2 \sin \theta.$$

$$= \tfrac{1}{2}r^2(\theta - \sin \theta).$$

EXERCISE XV (C)

1. Find the area of the segment of a circle, the arc of which makes an angle of 45° at the centre of a circle, the radius of which is 2 ft.

2. A chord 18 inches long is placed in a circle of radius 25 inches, find

(*i*) the angle subtended at the centre ;

(*ii*) the length of the arc ;

(*iii*) area of the segment and the sector respectively.

(*P.U. 1935*)

MISCELLANEOUS EXERCISES ON CHAPTER XV

1. a be the side of a regular polygon of n sides, R and r the radii of the circumscribed and inscribed circles, prove that

$$R + r = \frac{a}{2} . \cot \frac{\pi}{2n}.$$

2. The area of a regular polygon of m sides circumscribed about a circle is to the area of the circumscribed polygon of $2m$ sides as 3 : 2 ; find the value of m.

3. Of two regular polygons of n sides, one is circumscribed to and the other is inscribed in a given circle. Prove that the perimeters of the circumscribed polygon, the circle, and the inscribed polygon are in the ratio

$$\sec \frac{\pi}{n} : \operatorname{cosec} \frac{\pi}{n} : 1$$

and that the areas of the polygons are in the ratio $1 : \cos^2 \frac{\pi}{n}$.

4. Find the areas of the following polygons, if the circumradius of each of them be 2 feet.

(*i*) square, (*ii*) hexagon, (*iii*) octagon.

5. If S_1 be the area of a regular polygon of $2n$ sides inscribed in a circle and S_2, S_3 be the areas of inscribed and circumscribed regular polygons of n sides in and about the same circle, prove that $S_2S_3 = S_1^2$.

6. A regular polygon of $2n$ sides, n of which are equal to a, and n to b, is inscribed in a circle. Show that the radius of the circle is

$$\frac{1}{2}\left[a^2+b^2+2ab\cos\frac{\pi}{n}\right]^{\frac{1}{2}}\cdot\operatorname{cosec}\frac{\pi}{n}.$$

ANSWERS TO EXERCISES IN CHAPTER XV

Exercise XV (A)

1. $(\sqrt{6}+\sqrt{2})$ feet. 2. 6.

Exercise XV (B)

2. (*ii*) ·01745. 5. 1440 yards.

6. 200 ft.

Exercise XV (C)

1. $\left(\frac{\pi}{2}-\sqrt{2}\right)$ sq. 2. (*i*) $42°12'$; (*ii*) arc $= 18·4''$

(*iii*) 20·4 sq. m. ; 230·3 sq. inches.

Miscellaneous Exercises on Chapter XV

2. 3. 4. (*i*) 8 sq. feet. (*ii*) $6\sqrt{3}$ sq. ft. (*iii*) $8\sqrt{2}$ sq. ft.

CHAPTER XVI

INVERSE CIRCULAR FUNCTIONS AND SUMMATION OF TRIGONOMETRICAL SERIES

SECTION (A)

Inverse Circular Functions

16·1. The trigonometric equations $\sin\theta=a$, $\tan\theta=c$, etc., are satisfied by indefinitely many values of θ. The numerically smallest of these values of θ in the case of $\sin\theta=a$ is denoted by $\sin^{-1}a$ (read as inverse sin a) and in the case of $\tan\theta=c$ is denoted by $\tan^{-1}c$ (read as inverse tan c).

Similarly $\cos^{-1}(a)$, $\cot^{-1}(a)$, $\sec^{-1}(a)$ and $\text{cosec}^{-1}(a)$ are defined.

Def. The quantities $\sin^{-1}(a)$, $\cos^{-1}x$, $\tan^{-1}y$, etc. are called **Inverse Circular Functions.**

Note 1. $\text{Sin}^{-1}a$ and $(\sin a)^{-1}$ are two different quantities; the former is inverse sine a and the latter is $\dfrac{1}{\sin a}$.

Note 2. If there are two angles, which are numerically the smallest, one positive and the other negative, which satisfy the inverse equation, the positive angle is to be taken.

e.g. $\cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ has the smallest value as $45°$ or $-45°$.

But we shall reject the negative value, thus $\cos^{-}\left(\dfrac{1}{\sqrt{2}}\right)=45°$.

Note 3. From definition it is clear that $\sin\theta=a$ and $\theta=\sin^{-1}(a)$ are two identical equations. Thus $\sin^{-1}(a)$ is the angle the sine of which is equal to a. Similarly $\cot^{-1}(x)$ is the angle the contangent of which is equal to x.

16·2. Limits of the Inverse Circular Functions :—

From the discussion in the last article 16·1, it is clear that

(*i*) If a is +ve, $\sin^{-1}(a)$, $\cos^{-1}(a)$ and $\tan^{-1}(a)$ lie between $0°$ and $90°$.

(*ii*) If a is $-$ve

(*a*) $\sin^{-1}(a)$ and $\tan^{-1}(a)$ lie between -90° and 0°

(*b*) $\cos^{-1}(a)$ lies between 90° and 180°.

ILLUSTRATIONS

1. Find the values of $\sin^{-1}(\frac{1}{2})$ and $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

(*i*) By definition, we are to select the smallest out of all the angles, the sines of which are $=\frac{1}{2}$. The smallest value is 30° $\therefore$ $\sin^{-1}(\frac{1}{2})=30^\circ$.

(*ii*) By Art. 16·2 the value of $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ lies between -90° and 0°.

Thus we take the angle between $+90^\circ$ and 0° the sine of which is equal to $-\frac{1}{\sqrt{2}}$. That angle is -45° $\therefore$ $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ $=-45^\circ$.

2. Find the values of $\cos^{-1}(\frac{1}{2})$ and $\cos^{-1}(-\frac{1}{2})$

(*i*) By Art. 16·2 the values of $\cos^{-1}(\frac{1}{2})$ lies between 0° and 90°.

That angle is clearly 60° $\therefore$ $\cos^{-1}(\frac{1}{2})=60^\circ$.

(*ii*) By Art. 16·2, the value of $\cos^{-1}(-\frac{1}{2})$ lies between 90 and 180°.

That angle is clearly 120° $\therefore$ $\cos^{-1}(-\frac{1}{2})=120^\circ$.

3. Similarly it can be argued that

(*i*) $\tan^{-1}(\sqrt{3})=60^\circ$ and $\tan^{-1}(-1)=-45^\circ$.

SOLVED EXAMPLES

1. Prove that $\sin^{-1}\frac{4}{5}=\cos^{-1}(\frac{3}{5})=\tan^{-1}(\frac{4}{3})$.

Let $\sin^{-1}\frac{4}{5}=a$ $\therefore$ By def. $\sin a=\frac{4}{5}$ ... ... (*i*)

From (*i*) $\cos a=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$ ... ... (*ii*)

$\therefore$ $a=\cos^{-1}(\frac{3}{5})$.

Again from (*i*) and (*ii*) by div. $\tan a=\frac{4}{3}$ $\therefore$ $a=\tan^{-1}(\frac{4}{3})$.

$\therefore$ All these values are equal, as by supposing the first of them to be equal to a, we have proved that all of them are equal to a.

2. Prove that $\sin^{-1}(\frac{3}{5})+\cos^{-1}(\frac{12}{13})=\tan^{-1}(\frac{56}{33})$.

Let $\sin^{-1}(\frac{3}{5})=\alpha$ and $\cos^{-1}(\frac{12}{13})=\beta$.

$\therefore \sin\alpha=\frac{3}{5}$ ... ... (i)

and $\cos\beta=\frac{12}{13}$ ... ... (ii)

From (i) $\tan\alpha=\dfrac{\frac{3}{5}}{\sqrt{1-\frac{9}{25}}}=\frac{3}{4}$

and from (ii) $\tan\beta=\dfrac{\sqrt{1-\frac{144}{169}}}{\frac{12}{13}}=\frac{5}{12}$.

Now L.H.S. $=\alpha+\beta$ ... ... (iii)

Again $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha.\tan\beta}=\dfrac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}}=\dfrac{\frac{14}{12}}{\frac{11}{16}}=\frac{56}{33}$

$\therefore \tan^{-1}(\frac{56}{33})=\alpha+\beta$ ... ... (iv)

From (iii) and (iv)

$\sin^{-1}(\frac{3}{5})+\cos^{-1}(\frac{12}{13})=\tan^{-1}(\frac{56}{33})$.

EXERCISE XVI (A)

Find the values of

1. $\sin^{-1}\left(\dfrac{-\sqrt{3}}{2}\right)$. 2. $\cos^{-1}(\frac{1}{2})$

3. Prove that $\sin^{-1}(\frac{5}{13})=\cos^{-1}(\frac{12}{13})=\tan^{-1}(\frac{5}{12})$.

4. Prove that $2\sin^{-1}(\frac{3}{5})=\sin^{-1}(\frac{24}{25})$.

Sol. Put $\sin^{-1}(\frac{3}{5})=\alpha$. $\therefore \sin\alpha=\frac{3}{5}$.

Hence $\cos\alpha=\sqrt{1-\frac{9}{25}}=\frac{4}{5}$

Now L.H.S. $=2\alpha$ ... ... (i)

$\therefore$ we find $\sin 2\alpha$, which is equal to

$2\sin\alpha\cos\alpha$ or $2\cdot\frac{3}{5}\cdot\frac{4}{5}=\frac{24}{25}$.

$\therefore \sin 2\alpha=\frac{24}{25}$ or $2\alpha=\sin^{-1}(\frac{24}{25})$ ... ... (ii)

From (i) and (ii) $2\sin^{-1}(\frac{3}{5})=\sin^{-1}(\frac{24}{25})$.

5. Prove that $2\cos^{-1}(\frac{3}{5})=\cos^{-1}(-\frac{7}{25})$.

6. Prove that (i) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)=\tan^{-1}(x)$.

(ii) $\cos^{-1}(2x^2-1)=2\cos^{-1}(x)$.

7. Prove that $\sin^{-1}\dfrac{1}{\sqrt{5}}+\cot^{-1}(3)=\tan^{-1}(1)=\dfrac{\pi}{4}$.

Addition and Subtraction Formulae

16·3. To prove that

(*i*) $\sin^{-1}(x)+\sin^{-1}(y)=\sin^{-1}(x\sqrt{1-y^2}+y.\sqrt{1-x^2})$.

(*ii*) $\sin^{-1}(x)-\sin^{-1}(y)=\sin^{-1}(x\sqrt{1-y^2}-y.\sqrt{1-x^2})$.

Let $\sin^{-1}(x)=\theta$ and $\sin^{-1}(y)=\phi$ ... ... (*i*)

$\therefore\ \sin\theta=x$ and $\sin\phi=y$ ... ... (*ii*)

Hence $\cos\theta=\sqrt{1-x^2}$ and $\cos\phi=\sqrt{1-y^2}$ ... ... (*iii*)

Now $\sin(\theta+\phi)=\sin\theta.\cos\phi+\cos\theta\sin\phi$

$=x\sqrt{1-y^2}+y.\sqrt{1-x^2}$... from (*ii*) and (*iii*)

$\therefore\ \theta+\phi=\sin^{-1}(x.\sqrt{1-y^2}+y.\sqrt{1-x^2}$ ... ... (*iv*)

From (*i*) and (*iv*)

$\sin^{-1}(x)+\sin^{-1}(y)=\sin^{-1}(x\sqrt{1-y^2}+y.\sqrt{1-x^2})$

Similarly it can be proved that

$\sin^{-1}(x)-\sin^{-1}(y)=\sin^{-1}(x.\sqrt{1-y^2}-y.\sqrt{1-x^2})$.

16·4. To prove that

(*i*) $\cos^{-1}(x)+\cos^{-1}(y)=\cos^{-1}[xy-\sqrt{(1-x^2)(1-y^2)}]$

(*ii*) $\cos^{-1}(x)-\cos^{-1}(y)=\text{cas}^{-1}[xy+\sqrt{(1-x^2)(1-y^2)}]$.

Let $\cos^{-1}(x)=\theta$ and $\cos^{-1}(y)=\phi$ ... ... (*i*)

$\therefore\ \cos\theta=x$ and $\cos\phi=y$ ... ... (*ii*)

Hence $\sin\theta=\sqrt{1-x^2}$ and $\sin\phi=\sqrt{1-y^2}$ ... ... (*iii*)

Again $\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$

$=xy-\sqrt{(1-x^2)(1-y^2)}$ from (*ii*) and (*iii*)

$\therefore\ \theta+\phi=\cos^{-1}[xy-\sqrt{(1-x^2)(1-y^2)}]$.

From (*i*) and (*iv*)

$\cos^{-1}(x)+\cos^{-1}(y)=\cos^{-1}[xy-\sqrt{(1-x^2)(1-y^2)}]$.

Similarly by writing down the value of $\cos(\theta-\phi)$, we can prove that

$\cos^{-1}(x)-\cos^{-1}(y)=\cos^{-1}[xy+\sqrt{(1-x^2)(1-y^2)}]$.

16·5. To prove that

(*i*) $\tan^{-1}(x)+\tan^{-1}(y)=\tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

(*ii*) $\tan^{-1}(x)-\tan^{-1}(y)=\tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$.

Let $\qquad \tan^{-1}(x)=\theta \quad \text{and} \quad \tan^{-1}(y)=\phi \qquad \ldots \qquad \ldots \quad (i)$

$\therefore \qquad \tan\theta=x \quad \text{and} \quad \tan\phi=y \qquad \ldots \qquad \ldots \quad (ii)$

$$\text{Now } \tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

$$=\frac{x+y}{1-xy} \text{ from } (ii)$$

$$\therefore \quad \theta+\phi=\tan^{-1}\left(\frac{x+y}{1+xy}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \ (iii)$$

From (*i*) and (*iii*)

$$\tan^{-1}(x)+\tan^{-1}(y)=\tan^{-1}\left(\frac{x+y)}{1-xy}\right).$$

Similarly $\tan^{-1}(x)-\tan^{-1}(y)$ can be proved to be equal to

$$\tan^{-1}\left(\frac{x-y}{1+xy}\right).$$

SOLVED EXAMPLES

1. Prove that $\sin^{-1}(\frac{3}{5})+\sin^{-1}\frac{12}{13}=\sin^{-1}(-\frac{33}{65})$

$$\sin^{-1}\tfrac{3}{5}+\sin^{-1}\tfrac{12}{13}=\sin^{-1}\left[\tfrac{3}{5}\cdot\sqrt{1-\tfrac{12}{13}^2}-\tfrac{12}{13}\cdot\sqrt{1-\tfrac{9}{25}}\right]$$

$$\left[\because \sin^{-1}(x)+\sin^{-1}(y)=\sin^{-1}(x.\sqrt{1-y^2}+y.\sqrt{1-x^2}\right]$$

$$=\sin^{-1}(\tfrac{3}{5}\cdot\tfrac{5}{13})-\tfrac{12}{13}\cdot\tfrac{4}{5})$$

$$=\sin^{-1}(-\tfrac{33}{65}).$$

2. Prove that $\cos^{-1}(\frac{4}{5})+\cos^{-1}(\frac{5}{13})=\cos^{-1}(\frac{-16}{65})$

$$\cos^{-1}(\tfrac{4}{5})+\cos^{-1}(\tfrac{5}{13})=\cos^{-1}\left[\tfrac{4}{5}\cdot\tfrac{5}{13}-\sqrt{(1-\tfrac{16}{25})(1-\tfrac{25}{169})}\right]$$

$$\left[\because \cos^{-1}(x)+\cos^{-1}(y)=\cos^{-1}\{xy-\sqrt{(1-x^2)(1-y^2)}\}\right]$$

$$=\cos^{-1}(\tfrac{4}{13}-\tfrac{3}{5}\cdot\tfrac{12}{13})=\cos^{-1}(\tfrac{-16}{65}).$$

3. Prove that $\tan^{-1}\frac{1}{2}+\tan^{-1}(\frac{1}{3})=\dfrac{\pi}{4}$

$$\tan^{-1}(\tfrac{1}{2})+\tan^{-1}(\tfrac{1}{3})=\tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right)$$

$$\left[\because \tan^{-1}(x)+\tan^{-1}(y)=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\tan^{-1}(1)=\frac{\pi}{4}.$$

EXERCISE XVI (B)

Prove that

1. $\sin^{-1}(\frac{4}{5})+\sin^{-1}(\frac{5}{13})+\sin^{-1}(\frac{16}{65})=\frac{\pi}{2}$.

[Hint. Combine the first two terms and then combine this result with the third term.]

2. $\sin^{-1}\frac{3}{5}+\cos^{-1}(\frac{12}{13})=\sin^{-1}(\frac{56}{65})$.

[Hint. $\cos^{-1}(\frac{12}{13})=\sin^{-1}(\frac{5}{13})$.]

3. $\cos^{-1}(\frac{4}{5})+\cos^{-1}(\frac{12}{13})=\cos^{-1}(\frac{33}{65})$.

4. $\tan^{-1}(\frac{1}{2})+\tan^{-1}(\frac{1}{5})+\tan^{-1}(\frac{1}{8})=\frac{\pi}{4}$.

5. $\tan^{-1}(\frac{1}{7})+\tan^{-1}(\frac{1}{13})=\tan^{-1}(\frac{2}{9})$.

6. $\tan^{-1}(\frac{43}{66})-\tan^{-1}(\frac{3}{8})=\tan^{-1}(\frac{2}{9})$.

7. $\tan^{-1}(\frac{1}{3})+\tan^{-1}(\frac{1}{5})+\tan^{-1}(\frac{1}{7})+\tan^{-1}(\frac{1}{8})=\frac{\pi}{4}$.

8. $\tan^{-1}x+\tan^{-1}\frac{1-x}{1+x}=\frac{\pi}{4}$.

9. $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{\pi}{4}$.

10. $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

SECTION (B)

Summation of Trigonometrical Series

16 6. The methods employed for the summation of Trigonometric series will, in general, be similar to those employed for summing up series in Algebra. Most of the trigonometric series are summed up by reducing the series to (i) a Geometrical Progression, (ii) a Binomial series, etc. Quite often the method of differences will be found very useful.

16 61. To find the sum of the series

$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\sin[\alpha+(n-1)\beta]$.

Multiply each term by $2\sin\frac{\beta}{2}$, we get

[Note this step : the difference between the anglss is β and we multiply each term by twice sine of half the difference.]

$$2\sin\frac{\beta}{2}\cdot\sin\alpha=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{\beta}{2}\right)$$

[$\because$ 2 sin P. 2in Q = cos (P − Q) − cos (P + Q)].

$$2\sin\frac{\beta}{2}\cdot\sin(\alpha+\beta)=\cos\left(\alpha+\frac{\beta}{2}\right)-\cos\left(\alpha+\frac{3\beta}{2}\right).$$

$$2\sin\frac{\beta}{2}\cdot\sin(\alpha+2\beta)=\cos\left(\alpha+\frac{3\beta}{2}\right)-\cos\left(\alpha+\frac{5\beta}{2}\right).$$

....

$$2\sin\frac{\beta}{2}\cdot\sin[\alpha+(n-1)\beta]=\cos\left(\alpha-\overline{2n-3}\frac{\beta}{2}\right)-\cos\left(\alpha+\overline{2n-1}\frac{\beta}{2}\right)$$

Denoting S_n as the required sum, we have by addition

$$2\sin\frac{\beta}{2}\cdot S_n=2\sin\frac{\beta}{2}\,[\sin\alpha+\sin(\alpha+\beta)+\ldots+\sin\{\alpha+(n-1)\beta\}]$$

$$=\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+\overline{2n-1}\frac{\beta}{2}\right)$$

[$\because$ other terms cancel in pairs].

$$=2\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]\cdot\sin\frac{n\beta}{2}.$$

$$\therefore\quad S_n=\frac{\sin\left[\alpha+(n-1)\frac{\beta}{2}\right]\cdot\sin\frac{n\beta}{2}}{\sin\frac{n\beta}{2}}$$

Cor. Putting $\beta=\alpha$, we get

$$S_n=\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots\ldots+\sin n\alpha.$$

$$=\frac{\sin\frac{n+1}{2}\alpha\cdot\sin\frac{n\alpha}{2}}{\sin\frac{\beta}{2}}$$

SOLVED EXAMPLES

1. Show that

$$\sin\alpha+\sin(\alpha-\beta)+\sin(\alpha-2\beta)+\ldots\ldots+\sin(\alpha-\overline{n-1}\beta).$$

$$=\frac{\sin\left[\alpha-(n-1)\frac{\beta}{2}\right].\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

By putting $\beta=-\beta$ in S_n, we have

$$S_n=\frac{\sin\left[\alpha+(n-1)\frac{-\beta}{2}\right].\sin\frac{n(-\beta)}{1}}{\sin\frac{-\beta}{2}}$$

$$=\frac{\sin\left[\alpha-(n-1)\frac{\beta}{2}\right].\sin\frac{n\beta}{2}}{-\sin\frac{\beta}{2}}.$$

$$=\frac{\sin\left[\alpha-(n-1)\frac{\beta}{2}\right].\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

2. Sum the series to n terms

$\sin x+\sin 3x+\sin 5x+\ldots\ldots$

In the series

$S_n=\sin\alpha+\sin(\alpha+\beta)+\ldots\ldots+\sin[\alpha+(n-1)\beta].$

$$=\frac{\sin\left[\alpha+(n-1)\frac{\beta}{2}\right].\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}},$$ we put

$\alpha=x$ and $\beta=2x$, the series becomes

$S_n=\sin x+\sin 3x+\sin 5x+\ldots\ldots$to n ferms.

$$=\frac{\sin\left[x+(n-1)\frac{2x}{2}\right].\sin\left(\frac{n.2x}{2}\right)}{\sin\frac{2x}{2}}$$

$$=\frac{\sin(nx)\sin(nx)}{\sin x}=\frac{\sin^2(nx)}{\sin x}.$$

16·62. To find the sum of the series
$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots+\cos[\alpha+(n-1)\beta]$.

Multiply each of the terms by $2\sin\frac{\beta}{2}$, we get

$$2\sin\frac{\beta}{2}\cos\alpha=\sin\left(\alpha+\frac{\beta}{2}\right)-\sin\left(\alpha-\frac{\beta}{2}\right)$$

$[\because\ 2\sin P.\cos Q=\sin(P+Q)+\sin(P-Q)$ or
$=\sin(P+Q)-\sin(Q-P)]$

$$2\sin\frac{\beta}{2}.\cos(\alpha+\beta)=\sin\left(\alpha+\frac{3\beta}{2}\right)-\sin\left(\alpha+\frac{\beta}{2}\right)$$

$$2\sin\frac{\beta}{2}.\cos(\alpha+2\beta)=\sin\left(\alpha+\frac{5\beta}{2}\right)-\sin\left(\alpha+\frac{3\beta}{2}\right)$$

....

$$2\sin\frac{\beta}{2}.\cos[\alpha+(n-1)\beta)=\sin\left(\alpha+\frac{2n-1}{2}\beta\right)$$
$$-\sin\left(\alpha+\frac{2n-3}{2}\beta\right)$$

$\therefore$ By denoting the series by the C_n we get

$$2\sin\frac{\beta}{2}.C_n=2\sin\frac{\beta}{2}[\cos(\alpha)+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots$$
$$+\cos[\alpha+(n-1)\beta]$$
$$=\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin\left(\alpha-\frac{\beta}{2}\right)$$
$$=2\cos\left(\alpha+\frac{n-1}{2}\beta\right).\sin\frac{n\beta}{2}$$

$$\therefore\quad C_n=\frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right).\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

Cor. By putting $\beta=\alpha$, we get

$C_n=\cos\alpha+\cos2\alpha+\ldots\ldots\ldots+\cos n\alpha.$

$$=\frac{\cos\frac{n+1}{2}\alpha\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$$

Note 1. It should be observed that the sums S_n and C_n are respectively $\dfrac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$ times the sine of the average angle in C_n and the cosine of the average angle in C_n.

Note 2. The sum C_n can be derived from the sum S_n by putting $\alpha=\alpha+\frac{\pi}{2}$ for α in S_n.

Now $S_n=\sin(\alpha)+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots\ldots$
$+\sin(\alpha+\overline{n-1}\beta)$

$$=\frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\cdot\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

$\therefore\ \sin\left(\alpha+\frac{\pi}{2}\right)+\sin\left(\alpha+\frac{\pi}{2}+\beta\right)+\sin\left(\alpha+\frac{\pi}{2}+2\beta\right)+\ldots$
$+\sin\left(\alpha+\frac{\pi}{2}+\overline{n-1}\beta\right)$

$$=\frac{\sin\left(\alpha+\frac{\pi}{2}+\frac{n-1}{2}\beta\right)\cdot\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

Or $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots+\cos(\alpha+\overline{n-1}\beta)$

$$=\frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\cdot\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

Note 3. Similarly the sum S_n can be derived from the sum C_n by putting $\alpha=\alpha+\frac{\pi}{2}$ for α in C_n.

SOLVED EXAMPLES

1. Show that $\cos x+\cos(x-a)+\cos(x-2a)+\ldots\ldots+\cos[x-(n-1)a]$

$$=\frac{\cos\left[x-\frac{n-1}{2}a\right].\sin\frac{na}{2}}{\sin\frac{a}{2}}$$

In $C_n=\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots+\cos[\alpha+(n-1)\beta]$

$$=\frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right).\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}},$$ put $\alpha=x$ and $\beta=-a$,

we get $\cos x+\cos(x-a)+\cos(x-2a)+\ldots\ldots+\cos[a=(n-1)\beta]$

$$=\frac{\cos\left[x-\frac{n-1}{2}a\right].\sin\frac{na}{2}}{\sin\frac{a}{2}}$$

2. Find the sum to n terms for

$\cos x+\cos 3x+\cos 5x+\ldots\ldots$

In the sum $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots+\cos[\alpha+(n-1)\beta]$.

$$=\frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right).\sin\frac{n\beta}{2}}{\sin\frac{\beta}{3}},$$ put $\alpha=x$ and $\beta=2x$

we get $\cos x+\cos 3x+\cos 5x+\ldots\ldots n$ terms

$$=\frac{\cos\left(x+\frac{n-1}{2}.2x\right).\sin\frac{n.2x}{2}}{\sin\frac{2x}{2}}$$

$$=\frac{\cos nx\sin nx}{\sin x}.$$

EXERCISE XVI (C)

Sum to n terms the series

1. $\sin x+\sin(x-y)+\sin(x-2y)+\ldots\ldots$

2. $\sin x-\sin 2x+\sin 3x-\sin 4x+\ldots\ldots$

[**Hint**: Put $\pi-x$ for α in Cor. Art. 16·61].

3. $\cos x+\sin 2x-\cos 3x-\sin 4x+\cos 5x+\sin 6x+\ldots\ldots$

[**Hint**: Put $\frac{\pi}{2}-x$ for α in Cor. Art. 16·61]

4. Prove that $1+\cos\alpha+\cos 2\alpha+\ldots\ldots+\cos(n-1)\alpha$

$$=\frac{\cos\frac{n-1}{2}\alpha.\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$$

5. Sum to n terms the series

$$\sin x.\cos x+\sin 2x.\cos 2x+\sin 3x.\cos 3x+\ldots\ldots$$

[**Hint**: First term $=\frac{1}{2}\sin 2x$ etc.]

MISCELLANEOUS EXERCISES ON CHAPTER XVI

1. Prove that $\tan^{-1}\left(\frac{x-1}{x+1}\right)-\tan^{-1}(x)=-\frac{\pi}{4}$.

2. Show that $\tan^{-1}(x)+\tan^{-1}(y)+\tan^{-1}(z)$

$$=\tan^{-1}\left(\frac{x+y+z-xyz}{1-yz-zx-xy}\right)$$

[**Hint**: Put $\tan^{-1}(x)=A$, $\tan^{-1}(y)=B$, $\tan^{-1}(z)=C$

$\therefore\quad x=\tan A,\quad y=\tan B,\quad z=\tan C$

Now $\tan(A+B+C)$

$$=\frac{\tan A+\tan B+\tan C-\tan A.\tan B.\tan C}{1-\tan B.\tan C-\tan C.\tan A-\tan A.\tan B}$$

$$=\frac{x+y+z-xyz}{1-yz-zx+xy}\text{ etc.}$$

3. Prove that

$$\tan^{-1}1+\tan^{-1}2+\tan^{-1}3=\pi=2(\tan^{-1}1+\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{3}).$$

4. Prove that $\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{1}{3\sqrt{11}}+\sin^{-1}\frac{3}{\sqrt{11}}=\frac{\pi}{2}$.

5. Find the sum of the series up to n terms

(i) $\sin^2\alpha+\sin^2(2\alpha)+\sin^2(3\alpha)+\ldots\ldots$

(ii) $\cos^2\alpha+\cos^2(2\alpha)+\cos^2(3\alpha)+\ldots\ldots$

(iii) $\cos^3\alpha+\cos^3(2\alpha)+\cos^3(3\alpha)+\ldots\ldots$

(iv) $\sin^3\alpha+\sin^3(2\alpha)+\sin^3(3\alpha)+\ldots\ldots$

[Hint : (i) & (ii) Use $2\sin^2\alpha=1-\cos 2\alpha$ and $2\cos^2\alpha=1+\cos 2\alpha$.
(iii) and (iv) $4\cos^3\alpha=\cos 3\alpha+3\cos\alpha$ and $4\sin^3\alpha=3\sin\alpha-\sin 3\alpha$.

6. Sum to n terms the series

(i) $\cos x.\cos(2x)+\cos(2x)\cos(3x)+\ldots\ldots$

(ii) $\sin x\sin 2x+\sin 2x.\sin 3x+\sin 3x.\sin 4x+\ldots$

7. Prove that (i) $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$

(ii) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$

(iii) $3\tan^{-1}x=\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

ANSWERS TO EXERCISES IN CHAPTER XVI

Exercise XVI (a)

1. $-60°$. 1. $60°$.

Exercise XVI (c)

1. $\dfrac{\sin\left(x-\frac{n-1}{2}\cdot y\right).\sin\frac{ny}{2}}{\sin\frac{y}{2}}$ 2. $\dfrac{\sin\frac{n(\pi-x)}{2}.\sin\frac{(n+1)(\pi-x)}{2}}{\cos\frac{x}{2}}$

3. $\dfrac{\sin\frac{n}{2}\left(\frac{\pi}{2}-x\right).\sin\frac{n+1}{2}\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}$ 5. $\dfrac{\sin(n+1)x\sin nx}{2\sin x}$

Miscellaneous Exercise on Chapter XVI

5. (i) $\dfrac{\frac{n}{2}-\frac{1}{2}\sin nx.\cos(n+1)x}{\sin x}$

(ii) $\dfrac{\frac{n}{2}\cos(n+1)x.\sin nx}{2\sin x}$

(*iii*) $$\frac{\frac{3}{4}\sin\frac{nx}{2}\cos\frac{n+1}{2}x}{\sin\frac{x}{2}}+\frac{\frac{1}{4}\sin\frac{3nx}{2}\cdot\cos\frac{3(n+1)x}{2}}{\sin\frac{3x}{2}}$$

(*iv*) $$\frac{3\sin(n+1)\frac{x}{2}\cdot\sin\frac{nx}{2}}{4\sin\frac{x}{2}}-\frac{\sin\frac{3}{2}(n+1)x\cdot\sin\frac{3nx}{2}}{4\sin\frac{3x}{2}}$$

6. (*i*) $$\frac{n+1}{2}\cos x+\frac{1}{2}\frac{\sin 2(n+1)x}{\sin 2x}$$

(*ii*) $$\frac{n+1}{2}\cos x-\frac{1}{2}\frac{\sin 2(n+1)x}{\sin 2x}.$$

PANJAB UNIVERSITY PAPERS

1947

1. (*a*) Define $\tan\theta$ for all values of θ and prove that

$$\tan^2\theta = \sec^2\theta - 1$$

(*b*) If $6\cos^2\theta = 1$, find the other trigonometric ratios.

(*c*) The vertical angle of an isosceles triangle is $55°17'$, express the base angle in circular measure.

2. Establish the following :—

(1) $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

(2) $\sin 18° = \dfrac{\sqrt{5}-1}{4}$.

(3) $\sin 20° \sin 40° \sin 80° \sin 90° = \dfrac{\sqrt{3}}{8}$.

3. (*a*) Determine a general expression for all angles having the same sine.

Solve :—

$$\sin x + \cos 3x = \cos 5x.$$

(*b*) The angles of elevation of the top of a tower, as observed from the top and the foot of a tree 100 feet high, are 15° and 52° respectively. Find the height of the tower.

4. (*a*) Draw the graph of $\sin x$ as x varies from $-\pi$ to 2π and solve graphically.

$$\operatorname{cosec}^2 x = 1.$$

(*b*) Obtain the area of a circle of radius r with the help of

$$\operatorname*{Lt}_{\theta \to 0} \sin\theta = \theta.$$

5. In any triangle ABC, prove that

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

and deduce the value of $\cos \frac{B}{2}$ in terms of the sides.

Given sides $a = 31$.
$b = 42$,
$c = 57$;

Solxe the triangle

6. Prove that the area of a triangle ABC is given by

$$\Delta = \tfrac{1}{2}\, bc \sin A$$

and the circum-radius R is given by

$$R = \frac{a}{2 \sin A} = \frac{abc}{4\ \Delta}.$$

If the sides of a triangle are 45, 55, 70, calculate the area of the circumscribed circle.

1948

1. (*a*) Define a 'radian' and show that it is a constant angle.

(*b*) Prove that the circular measure of an angle at the centre of a circle is expressed by the fraction

$$\frac{\text{Subtending arc}}{\text{radius}}$$

(*c*) Show that the number of seconds in a radians is 206265 (approximately).

(*d*) The sun subtends an angle $32'$ at the eye and its distance is 93000000 miles, find the diameter of the sun in miles.

2. (*a*) Prove that $\sin (90 + \theta) = \cos \theta$ (for any valne of θ).

(*b*) Show that (1) $4 \cos \theta \cos (60 - \theta).\cos (60 + \theta) = \cos 3\theta$.
(2) $\sin^6\theta = 1 - 3 \cos^2\theta + 3 \cos^4\theta - \cos^6\theta$.

3. (*a*) Find a general expression for all angles having the same tangent.

(*b*) Find the limiting value of $\frac{\sin \theta}{\theta}$ when θ tends to zero.

4. (*a*) Draw the graph of $y = \tan x$, where x lies between 0° and 360°.

(b) In any triangle ABC, prove that

$$\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$$

where the letters have their usual significance.

5. (a) Prove that in any triangle ABC

$$a^2=b^2+c^2-2bc\cos A$$

$$\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

(b) Using logarithmeic tables, solve the triangle whose sides are 31·9, 56·31, 40·27.

6. (a) Solve the eqnation $\sin 5\theta = 16\sin^5\theta$.

(b) The angle of elevation of a tower from a point A due south of it is α and from a point B due east of A is β. If AB=C, show that the height of the tower is

$$\frac{C}{\sqrt{\cot^2\beta-\cot^2\alpha}}.$$

1949

1. (a) An arc of a circle with centre O is 7·7 inches in length and the angle AOB is 2·2 radians. Taking π to be $\frac{22}{7}$, calculate

(i) the radius of the circle,

(ii) the distance from O of the point of intersection of the tangents at A and B.

(b) The difference between two angles is 1° and the circular measure of their sum is 1; express each angle in grades.

2. Prove that $\cos\left(\theta+\frac{\pi}{2}\right)=-\sin\theta$, taking θ to be an angle lying between $\frac{\pi}{2}$ and π and using a dirgram appropriate for such a value of θ.

(b) A vertical tower stands on a horizontal plane, and is surmounted by a vertical flagstaff of height h. At a point on the plane the angle of elevation of the bottom of the flagstaff is α and that of the top of the flagstaff is β. Prove that the height of the tower is $\frac{h\tan}{\tan\beta-\tan\alpha}$.

3. (*a*) Draw the graph $y=\cos x$, when x lies between $-90°$ and $+180°$. Solve graphically cost $x=\frac{1}{2}$.

(*b*) Give the general value of x which will satisfy the equation $2\cos^2 x+3\sqrt{}\sin x-5=0$.

4. (*a*) Prove geometrically that

$$\sin(A-B)=\sin A\cdot\cos B-\cos A\cdot\sin B.$$

(*b*) In any triangle, prove that

(*i*) $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cdot\cot\frac{A}{2}$.

(*ii*) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$.

5. In any triangle ABC, prove that

(*i*) $r_1=s\tan\frac{A}{2}$.

(*ii*) $r_1+r_2+r_3-r=4R$ (the letters have their usual significance). Obtain $\sin 2x$ and $\cos 2x$ in terms of $\tan x$.

6. (*a*) Find the area of a triangle in terms of its sides. If $b=3$, $c=3\sqrt{3}$ and $B=30°$, find the third side and the other angles.

(*b*) Prove that

$$\cos 20°\cdot\cos 40°\cdot\cos 60°\cdot\cos 80°=\tfrac{1}{16}.$$

1950

1. Find the length of the arc which subtends an angle of 18° at the centre of a circle whose circumference is 21 ft.

(*b*) Express in circular measure the angle of 56°21′30″ to three places of decimals. $(\pi=\frac{22}{7})$.

(*c*) Assuming that the earth moves round the Sun in a circle of radius 95×10^6 miles, find the speed of the Earth per second to the nearest mile. $(\pi=3{\cdot}1416)$ taking the year as 365 days.

2. (*a*) Prove that

(*i*) $\sec^6\theta-\tan^6\theta=1+3\tan^2\theta\sec^2\theta$.

(*ii*) $(\sec\phi-\cos\phi)(\operatorname{cosec}\phi-\sin\phi=\frac{\tan\phi}{1+\tan^2\phi}$.

(*b*) On walking 150 ft. towards a chimney in a horizontal line through its base, the angular elevation of its top changes

from 30° to 60°; determine its height correct to one place of decimal.

3. (*a*) Solve the equation $8 \sin^2\theta - 2 \cos \theta = 5$.

(*b*) Draw the graph of $y = \tan \theta$, when θ lies between -90° and 90°.

4. (*a*) Prove geometrically that

$$\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

when A and B are positive and acute angles.

(*b*) Find $\tan 75^\circ$ correct to two places of decimals.

(*c*) Prove that $\dfrac{\sin 2A}{\sin A} - \dfrac{\cos 2A}{\cos A} = \sec A$.

5. (*a*) In any triangle, show that

(*i*) $\tan \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$.

(*ii*) $c^2 = (a+b)^2 \sin^2 \dfrac{C}{2} + (a-b)^2 \cos^2 \dfrac{C}{2}$.

(*b*) The sides of a triangle are 34, 20, 42; find the radii of its escribed circle.

6. In a triangle $a = 456{\cdot}12$, $b = 296{\cdot}86$, $C = 74^\circ 20'$, find A and B (use logarithmic tables).

1951

1. (*a*) Define a radian and find approximately its value in degrees, minutes and seconds.

(*b*) Find in radians and degrees the angle between the minute-hand and the hour-hand of a clock at twenty minutes to six.

2. (*a*) Show that for all values of θ, $\tan (\pi + \theta) = \tan \theta$.

(*b*) Prove the following identities:

(*i*) $\cot\left(\dfrac{\pi}{4} + \theta\right) \cot\left(\dfrac{\pi}{4} - \theta\right) = 1$.

(*ii*) $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1 + \sin A}{\cos A}$.

3. (*a*) Trace the changes in the values of $\cos \theta$, as θ varies from $-\pi$ to π. Illustrate these variations by means of a graph.

(b) A, B, C are points in succession on a straight level road and P is another point on the road so situated that the angles PAB, PBA, PCA are respectively 90°, 60°, and 45°. If a man walks at a uniform rate from A to B in 25 seconds, find how long it will take him, at the same rate, to walk from B to C.

4. (a) Prove that the $\log_a m = \log_b m \times \log_a b$ and find the value of $\log_2 3$ correct to 3 decimal places.

(b) Find the number of digits in 3^{62}.

5 In a triangle ABC, CA=6 feet, AB=5 feet and angle B=63°, eolve the triangle. (Use logarithmic tables).

6. (a) Find the radius of the escribed circle of a triangle ABC corresponding to the angle B.

(b) With usual notation prove that :

(i) $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

(ii) $r_1 + r_2 + r_3 = r + 4R$.

1952

1. (a) Express the value of a radian in degrees, minutes and seconds.

(b) Find in degrees, minutes and seconds, the angle subtended at the centre of a circle of diameter 10 feet by an arc 9 inches long.

(c) If $\tan\theta + \sec\theta = 4$, find $\sin\theta$ and $\cos\theta$.

2. (a) Prove that $\sec(90° + \theta) = -\operatorname{cosec}\theta$, taking θ to be an angle between $\frac{\pi}{2}$ and π and using a diagram appropriate for such a value of θ.

(b) Show that $\tan 225° \cot 405° + \tan 765° \cot 675° = 0$

(c) Find angles lying between 0 and 2π which satisfy the equation $4\sin^2\theta - 8\cos\theta + 1 = 0$.

3. (a) Show that $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$.

(b) Prove that $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$.

(c) If $A+B+C=\pi$, prove that

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

4. (*a*) Draw the graph of $\sec x$, as x changes from 0 to 2π.

(*b*) A man on a cliff observes a boat at an angle of depression of 30°, which is making for the shore immediately beneath him Three minutes later the angle of depression of the boat is 60°. How soon will it reach the shore ?

5. (*a*) In any triangle, prove that

$$\tan\frac{B-C}{C}=\frac{b-c}{b+c}\cot\frac{A}{2}.$$

(*b*) Two sides of a triangle are $\sqrt{3}+1$ and $\sqrt{3}-1$ and the included angle is 60°. Find the third side and the other 2 angles.

(*c*) Find the number of digits in 7^{31}.

6. (*a*) With usnal notation prove that :

(*i*) $r=(s-a)\tan\frac{A}{2}$.

(*ii*) $r_1=4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

(*b*) Prove that in an equilateral triangle.

$$r : R : r_1=1 : 2 : 3.$$

1953

1. (*a*) Prove $\sin^2\theta+\cos^2\theta=1$, where θ is any angle.

(*b*) If $8\sin A-\cos A=8\operatorname{cosec} A-\sec A$, prove

$\tan A=2$.

(*c*) From the top of a cliff of height h, the angle of depression of a boat on water below is θ, and the angle of elevation of a balloon, which is directly over the boat, is ϕ, prove that the height of the balloon above water is

$$h(1+\cot\theta\tan\phi).$$

2. (*a*) Name the units in the circular and the sexagesimal system of measurement of angle. and prove π radians = 2 right angles.

(*b*) The angles of a triangle are in AP. If number of the degrees in the least angle is to the number of radians in the largest angle as $45 : \frac{2\pi}{3}$, find the angles in degrees.

(*c*) Prove $\text{Lim} \frac{\sin\theta}{\theta}$ as $\theta \to 0$ is 1, and hence find, correct to 5 decimal places, the value of $\sin 10''$.

3. (*a*) If A and B are acute angles, whose sum is less than 90°, prove geometrically,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

(*b*) Prove $\frac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \tan 56^\circ$.

(*c*) Either find all angles between 0 and 360°, for which $\tan^2\theta = 1$, and give the general solution of

$$2\sin^2\theta = 1 + \cos\theta$$

or Simplify

$$\frac{\sin^2(180^\circ+\theta) \cdot \tan(270^\circ+\theta)}{\cot^2\left(\frac{3\pi}{2}\right) \cdot \cos^2(\pi-\theta)\ \text{cosec}\ (2\pi-\theta)}$$

4. In any triangle, prove

(*a*) $\cos\frac{A}{2}\sqrt{\frac{s(s-a)}{bc}}$.

(*b*) $(b^2-c^2)\sin 2A = a^2(\sin 2C - \sin 2B)$.

(*c*) $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$.

5. (*a*) Find circumradius of a $\triangle ABC$ with sides 5, 12, 13.

(*b*) With usual notation, prove $r_3 = \frac{\Delta}{s-c}$.

(*c*) Show $\sqrt{r\, r_1 r_2 r_3} = \Delta = \frac{1}{2}bc \sin A$.

6. (*a*) Graphically solve $\sin x = \cos x$ for values of x between -180° and $+180^\circ$.

(*b*) Given $\log_{10}2 = 0{\cdot}3010$, find, without using tables,

$$\log_{10}\sqrt[3]{{\cdot}0125}.$$

(*c*) Solve $\triangle$ ABC if $b = 56{\cdot}8$, $A = 79^\circ\ 31'$, $B = 44^\circ\ 24'$.

1954

Q. 1. (*a*) Define sin A and tan A for an angle of any magnitude.

(*b*) Show that $\sin\theta < \theta < \tan\theta$ for any positive angle less than a right angle.

Deduce $\underset{\theta\to 0}{\mathrm{Lt}}\dfrac{\sin\theta}{\theta}=1$

Q. 2. (*a*) Prove for all angles that $\tan(180+\theta)=\tan\theta$.

(*b*) Represent $y=\tan x$ graphically as x changes from 0 to π. What characteristics of the tangent are indicated by the graph?

What shape will the graph have as x changes from π to 2π?

Q. 3. (*a*) Prove geometrically that

$$\sin(A-B)=\sin A\cos B-\cos A\sin B.$$

(*b*) Prove that $\tan^{-1}a+\tan^{-1}b=\tan^{-1}\dfrac{a+b}{1-ab}$.

(*c*) Solve $\tan^{-1}3x+\tan^{-1}2x=\dfrac{\pi}{4}$.

Q. 4. (*a*) Find the value of θ between 90° and 180° for which $\sin\theta=\sin 40$.

(*b*) Obtain the general expression for all angles whose sine is equal to $\sin\alpha$.

(*c*) Solve the equation $3\sec^2\theta=2\operatorname{cosec}\theta$.

Q. 5. (*a*) Prove in any triangle that the sides are proportional to the sides of the opposite angles.

(*b*) In a triangle ABC, prove that

$$C\cos\frac{A-B}{2}=(a+b)\sin\frac{C}{2}.$$

(*c*) Find the shorter side of the triangle with 7 feet as base and $129^\circ 23'$, $38^\circ 36'$ as the angles at the base.

Q. 6. (*a*) Prove that the length of the inradius of a triangle $=\dfrac{\Delta}{s}$.

(*b*) Find the area of the circle of the triangle whose sides are 7, 8, 9 inches.

(*c*) In a triangle, prove that

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}=\frac{a^2+b^2+c^2}{\Delta}$$

Q. 7. (*a*) Prove that $\log_a m^r = r \log_a m$.

(*b*) Find the number of digits in 2^{16} given

$$\log_{10}2 = \cdot 30103$$

(*c*) Sum to *n* terms, the series

$$\sin A+\sin (A+B)+\sin (A+2B)+\ldots n \text{ terms.}$$

1955

Q. 1. (*a*) Define cos A for an angle of any magnitude.

(*b*) Trace the changes in magnitude and sign of cos A as A changes from 0 to 360°.

(*c*) Draw the graph of $y=\cos x$ as x varies from 0 to 180°. Indicate the shape of the cosine graph between $-180°$ and 0 without actually drawing it.

Q. 2. (*a*) Prove that

(*i*) $\sin (A+B) \sin (A-B)=\sin^2 A-\sin^2 B$.

(*ii*) $\cos^{-1}x+\cos^{-1}y=\cos^{-1}\{xy-\sqrt{(1-x^2)(1-y^2)}\}$;

(*b*) If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, prove that

$$x^2+y^2+z^2+2xyz=1.$$

Q. 3. (*a*) Prove that the area of a triangle ABC

$$=\sqrt{s(s-a)(s-b)(s-c)}$$

(*b*) Show that the radius of the *e* circle opposite to A in the above triangle

$$=\frac{\Delta}{s-a}.$$

(*c*) Prove that $rr_1+r_2r_3=bc$.

Q. 4. (*a*) Obtain the value of A between π and $\frac{3\pi}{2}$ for which $\cos A=-\frac{\sqrt{3}}{2}$.

(*b*) Find the general expression for all angles whose cosine is equal to cos a.

(*c*) Solve cosec $x=\cot x+\sqrt{3}$.

Q. 5 (*a*) In a triangle prove that

(*i*) $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$.

(*ii*) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$.

(*b*) Given $a=\sqrt{3}+1$

$b=2$.

$c=60^\circ$ solve the triangle.

Q. 6. (*a*) Show that with the usual notation, that in a circle $\frac{l}{r}=\theta^c$.

(*b*) Deduce the length of the circumference of a circle of radius r.

(*c*) The circumference of a circle of radius r is divided into five parts in A. P., such that the greatest part is 6 times the least. Find in radians the magnitude of the angle subtended at the centre by the least part.

Q. 7. (*a*) Prove that $\log_a{}^{mn}=\log_a{}^m+\log_a{}^n$.

(*b*) Given $a=345\cdot7$, $b=293\cdot9$, find the value of $\sqrt{a^2-b^2}$ by the use of logarithms.

(*c*) Sum the series

$$\sin 2A+\sin 5A+\sin 8A+\ldots\ldots\ldots\sin 23A$$

by the method of differences.

1956

Q. 1. (*a*) Prove that for all angles $\sin(180+\theta)=-\sin\theta$.

(*b*) Find the value of

$$\frac{\sin(-\theta)}{\cos(90+\theta)}-\frac{\tan(180-\theta)}{\cot(270-\theta)}+\frac{\operatorname{cosec}(360+\theta)}{\sec(90+\theta)}$$

(*c*) Prove that $\frac{1}{\sec\theta+\tan\theta}=\frac{1-\sin\theta}{\cos\theta}$.

Q. 2. (*a*) Prove Geometrically that

$$\cos(A+B)=\cos A\cos B-\sin A\sin B.$$

(*b*) Prove that

$$\sin A\cos(B+C)-\sin B\cos(C+A)$$
$$=\sin(A-B)\cos C.$$

(*c*) Solve $\tan^{-1}9x+\tan^{-1}6x=\frac{\pi}{4}$.

Q 3. (*a*) Draw the graph of $y=\sin x$ from 0 to 2π.

(*b*) Solve graphically $\sin x=x$.

(*c*) Solve the equation $\sec x+\cos x+\tan x=0$.

Q. 4. Prove in any triangle that :—

(*a*) $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$.

(*b*) $a\sin\left(\frac{A}{2}+C\right)=(b+c)\sin\frac{A}{2}$.

(*c*) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$.

Q. 5. (*a*) Define logarithms. Show that the mantissa of a number in common logarithms remains unchanged if the position of decimal point in the number is changed.

(*b*) Prove that $\log_a\frac{m}{n}=\log_a m-\log_a n$.

(*c*) Obtain the value of $\frac{\sqrt{24}\times\sqrt[3]{35}}{\sqrt[5]{53}}$.

Q. 6. (*a*) Prove that the area of a regular polygon of n sides inscribed in a circle of radius R

$$=\frac{n}{2}R^2\sin\frac{2\pi}{n}.$$

Deduce the area of a circle of radius R.

(*b*) Find the ratio of the areas of a regular hexagon and an equilateral triangle when their perimeters are equal.

Q. 7. (*a*) Find the angles of a triangle whose sides are 215, 275, 310.

(*b*) Sum the series by the method of differences

$$\cos\frac{A}{2}+\cos\frac{3A}{2}+\cos\frac{5A}{2}+\ldots\ldots\cos\frac{51A}{2}.$$

1957

Q. 1. (*a*) Define $\tan\theta$ for all values of θ and prove that

$$1+\tan^2\theta=\sec^2\theta.$$

(*b*) Given $\tan\theta=t$, express all other trigonometrical ratios in terms of t.

(*c*) Show that $\dfrac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}=\dfrac{1+\sin\theta}{\cos\theta}$.

Q. 2. (*a*) Prove geometrically that

$$\cos(A-B)=\cos A\cos B+\sin A\sin B.$$

(*b*) Prove that $\sin 18°=\dfrac{\sqrt{5}-1}{4}$.

(*c*) Prove that $\cos 20°\cos 40°\cos 60°\cos 80°=\frac{1}{16}$.

Q. 3. (*a*) Draw the graph of $y=\cos x$ when x lies between 0 and 2π.

Use the graph to solve $\cos x=x$.

(*b*) Prove that $\cot(180-\theta)=-\cot\theta$.

Q. 4. In any triangle ABC, prove that

(*a*) $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$.

(*b*) $R=\dfrac{abc}{4\Delta}$.

(*c*) $r_1r_2+r_2r_3+r_3r_1=s^2$.

Q. 5. (*a*) Find a general expression for angles having the same cosine.

(*b*) Solve $\sin 7\theta-\sin\theta=\sin 3\theta$.

(*c*) Prove that $\log_a m^n=n\log_a m$.

Q. 6. (*a*) Given $b=3{\cdot}15$, $c=1{\cdot}74$ and $A=37^\circ\ 20'$, solve the triangle. Also find the area of the triangle.

(*b*) Sum the series by the method of differences

$\cos A+\cos(A+B)+\cos(A+2B)+\cos(A+3B)+\ldots\ldots$ to n terms.

Q. 7. (*a*) If θ be the circular measure of an acute angle show that

$$\sin\theta<\theta<\tan\theta$$

Hence prove that

$$\operatorname*{Lt}_{\theta\to 0}\frac{\sin\theta}{\theta}=1.$$

(*b*) If R and r be the radii of the circumscribed and inscribed circles of a regular polygon of n sides each equal to a, show that

$$R+r=\frac{a}{2}\cot\frac{\pi}{3n}.$$

ANSWERS TO PANJAB UNIVERSITY PAPERS

1947

1. (*a*) $\sin\theta = \pm\sqrt{\frac{5}{6}}$, $\tan\theta = \pm\sqrt{5}$, $\cot\theta = \pm\frac{1}{\sqrt{5}}$,

$\sec\theta = \pm\sqrt{6}$, $\operatorname{cosec}\theta = \pm\sqrt{\frac{6}{5}}$. (*b*) $\frac{7483}{21600}\pi$ radians.

3. (*a*) (*ii*) $n\pi$ or $\frac{1}{4}\left[n\pi + (-1)^n.\frac{\pi}{6}\right]$. (*b*) 126·47.

4. (*a*) $x = \pm\frac{\pi}{2}$. 5. 101°36′, 46°12′, 32°12′.

6. 3854 sq. units nearly.

1948

1. (*iv*) 866,000 miles approximately.

5. (*ii*) 44°24′ (nearly) 34°42′ (nearly) 101°54′ (nearly).

6. (*i*) $\theta = n\pi$, $n\pi + (-1)^n\frac{\pi}{6}$, $n\pi + (-1)^n.\frac{7\pi}{6}$.

1949

1. (*i*) (*a*) $3\frac{1}{2}''$, (*b*) 7·71″ (*ii*) $\left(\frac{100}{\pi} + \frac{5}{9}\right)^\theta$, $\left(\frac{100}{\pi} - \frac{5}{9}\right)^\theta$.

3. (*i*) 70°30′ (nearly) (*ii*) $n\pi + (-1)^n\frac{\pi}{3}$.

6. (*i*) $a = b$, $c = 60°$, $A = 90°$.

1950

1. (*a*) 1·05 ft. (*b*) ·984 radians. (*c*) 19 m p s. (nearly).

2. (*b*) 129·9 ft. 3. $2n\pi \pm \frac{\pi}{3}$, $2n\pi \pm \cos^{-1}(-\frac{3}{4})$.

4. (*b*) 3·73. 5. (*b*) 12, 24, 56.

6. A = 68°25′ B = 37°15′.

1951

1. (a) $57°17'44{\cdot}8''$. (b) $170°$, $\frac{7\pi}{18}$.

3. (b) $18{\cdot}3$ seconds. 4. (b) 30.

5. $C=47°\,57'$
$A=69°3'$
$a=6{\cdot}29$.

1952

1. (a) $57°\,17'44{\cdot}8''$. (b) $8°35'$.
(c) $\sin\theta=\frac{15}{17}, -1$; $\cos\theta=\frac{8}{15}, 0$.

2. (c) $\theta=60°$. 4. (b) 6 minutes. 5. (b) $\sqrt{6}$, $15°$; $105°$.

1953

2. (b) $32°\,\frac{8}{11}$, $60°$, $87°\,\frac{3}{11}$.
(c) $={\cdot}000487$.

3. (c) $45°$, $225°$, $135°$, $315°$.
(c) $2n\pi\pm\frac{\pi}{3}$; $(2n\pm)\,\pi$ or $\cos\theta$

5. (a) $6{\cdot}5$.

6. (a) $x=45°$.
(b) $1{\cdot}3657$.
(c) $C=56°\,5'$; $a=79{\cdot}82$.

1954

3. (c) $x=-1$ and $x=\frac{1}{8}$.

4. (a) $\theta=140°$.
(c) $\text{Sin}\,\theta=\frac{1}{2}, -2$.

5. (c) $C=20{\cdot}97$ (from Antilog Tables).

6. (b) $25{\cdot}7080$.

7. (c) $$S=\frac{\text{Sin}\left[A+\frac{(\eta-1)R}{2}\right]\sin\frac{\eta\beta}{2}}{\sin\frac{\beta}{2}}.$$

1955

4. (*a*) $210°$. (*b*) $x = n\pi + \frac{2}{3}\pi$.

5. $A = 75°$, $B = 45°$, $C = \sqrt{6}$.

6. (*c*) $\dfrac{4\pi}{35}$. 7. (*a*) 197·2.

7. (*b*) $S = \dfrac{\sin 25A \sin 24A}{\sin \dfrac{3A}{2}}$.

1956

1. (*b*) 1. 2. (*c*) $x = \frac{1}{18}, -\frac{1}{3}$.

3. (*c*) $x = n\pi + (-1)^{n+1} \dfrac{\pi}{2}$.

5. (*c*) 7.244. 6. req. ratio is 2 : 1.

7. (*a*) $A = 42° 36'$, $B = 56° 46'$.

(*b*) $S = \dfrac{\sin 26A}{2 \sin \dfrac{A}{2}}$.

1957

6. (*a*) $B = 145° 37'$, $C = 27° 3'$. $a = 2·467$.
$\Delta = 3·538$.

TABLES

OF

LOGARITHMS AND ANTILOGARITHMS,
NATURAL SINES, COSINES AND TANGENTS,
LOGARITHMS OF SINES, COSINES AND TANGENTS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						5	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	36
11	0414	0453	0492	0531	0569						4	8	12	16	20	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	18	22	26	29	33
12	0792	0828	0864	0899	0934						3	7	11	14	18	21	25	28	32
						0969	1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	6	10	13	16	19	23	26	29
						1303	1335	1367	1399	1430	3	7	10	13	16	19	22	25	29
14	1461	1492	1523	1553	1584						3	6	9	12	15	19	22	25	28
						1614	1644	1673	1703	1732	3	6	9	12	14	17	20	23	26
15	1761	1790	1818	1847	1875						3	6	9	11	14	17	20	23	26
						1903	1931	1959	1987	2014	3	6	8	11	14	17	19	22	25
16	2041	2068	2095	2122	2148						3	6	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	16	18	21	23
17	2304	2330	2355	2380	2405						3	5	8	10	13	15	18	20	23
						2430	2455	2480	2504	2529	3	5	8	10	12	15	17	20	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	17	19	21
						2672	2695	2718	2742	2765	2	4	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2229	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

	0	1	2	3	4	5	6	7	8	9	1	2	3	[illegible]	[illegible]	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	[illegible]	[illegible]	[illegible]	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	[illegible]	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	[illegible]	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	[illegible]	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	[illegible]	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	[illegible]	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	[illegible]	3	4	5	6	7	8	9
·66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	[illegible]	[illegible]	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	[illegible]	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	[illegible]	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	[illegible]	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	[illegible]	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	[illegible]	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	[illegible]	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	[illegible]	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	15
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	12	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2232	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	14
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	10	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	7547	7558	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	9998	9999	9999	9999	9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0
90	1·000														

NATURAL COSINES
(*Subtract Mean Differences*)

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	[illegible]	3	4	5
0	1·000	1·000	1·000	1·000	1·000	1·000	9999	9999	9999	9999	0	0	0	0	0
1	9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	9994	9993	9993	9992	9991	9990	9990	9989	9988	9987	0	0	0	1	1
3	9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	2
6	9945	9943	9942	9940	9938	9930	9934	9932	9930	9928	0	1	1	1	2
7	·9925	9923	9021	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	3	4
18	9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20	9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	8746	8738	8729	8721	8712	8704	8695	8086	8678	8669	1	3	4	6	7
30	8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	·8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	·7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40	·7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	7431	7420	7408	7390	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	·7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

NATURAL COSINES

(Subtract Mean Differences)

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1	2	3	4	5
45	7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50	6428	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60	5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	4848	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	14
67	3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	14
68	3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70	3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80	·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	12	14
81	1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	15
86	0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15
90	0000														

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	[illegible]	3	4	5
0	0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	[illegible]	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	[illegible]	13	16
16	2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	[illegible]	13	16
17	3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	[illegible]	13	16
19	3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	[illegible]	13	17
22	4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	[illegible]	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	[illegible]	18	23
37	7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	1·0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1·0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1·0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1·1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1·1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1·1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1·2349	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1·2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1·3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1·3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1·4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1·4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1·5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1·6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1·6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1·7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1·8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1·8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1·9626	9711	9797	9883	9970	2·0057	2·0145	2·0233	2·0323	2·0413	15	29	44	58	73
64	2·0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	79
65	2·1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2·2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2·3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2·4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2·6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2·7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2·9042	9208	9375	9544	9714	9887	3·0061	3·0237	3·0415	3·0595	29	58	87	116	145
72	3·0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3·2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3·4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3·7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4·0108	0408	0713	1022	1335	1653	1976	2303	2635	2972	53	107	160	213	267
77	4·3315	3662	4015	4374	4737	5107	5483	5864	6252	6646					
78	4·7046	7453	7867	8288	8716	9152	9594	5·0045	5·0504	5·0970	Mean differences cease to be sufficiently accurate.				
79	5·1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80	5·6713	7297	7894	8502	9124	9758	6·0405	6·1066	6·1742	6·2432					
81	6·3138	3859	4596	5350	6122	6912	7720	8548	9395	7·0264					
82	7·1154	2066	3002	3962	4947	5958	6996	8062	9158	8·0285					
83	8·1443	2636	3863	5126	6427	7769	9152	9·0579	9·2052	9·3572					
84	9·5144	9·677	9·845	10·02	10·20	10·39	10·58	10·78	10·99	11·20					
85	11·43	11·66	11·91	12·16	12·43	12·71	13·00	13·30	13·62	13·95					
86	14·30	14·67	15·06	15·46	15·89	16·35	16·83	17·34	17·89	18·46					
87	19·08	19·74	20·45	21·20	22·02	22·90	23·86	24·90	26·03	27·27					
88	28·64	30·14	31·82	33·69	35·80	38·19	40·92	44·07	47·74	52·08					
89	57·29	63·66	71·62	81·85	95·49	114·6	143·2	191·0	286·5	573·0					
90	∞														

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	∞	$\bar{3}$·2419	$\bar{3}$·5429	7190	8439	9408	$\bar{2}$·0200	$\bar{2}$·0870	$\bar{2}$·1450	$\bar{2}$·1961					
1	$\bar{2}$·2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	$\bar{2}$·5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	$\bar{2}$·7188	7330	7468	7602	7731	7857	7979	8098	8213	8326					
4	$\bar{2}$·8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	$\bar{2}$·9403	9489	9573	9655	9736	9816	9894	9970	$\bar{1}$·0046	$\bar{1}$·0120	13	26	39	52	65
6	$\bar{1}$·0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	$\bar{1}$·0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	$\bar{1}$·1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	$\bar{1}$·1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10	$\bar{1}$·2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	$\bar{1}$·2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	$\bar{1}$·3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	$\bar{1}$·3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	$\bar{1}$·3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	$\bar{1}$·4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	$\bar{1}$·4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	$\bar{1}$·4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	$\bar{1}$·4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	$\bar{1}$·5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20	$\bar{1}$·5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	$\bar{1}$·5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	$\bar{1}$·5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	$\bar{1}$·5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	$\bar{1}$·6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	$\bar{1}$·6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	$\bar{1}$·6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	$\bar{1}$·6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	$\bar{1}$·6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	$\bar{1}$·6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	[illegible]	4	7	9	11
30	$\bar{1}$·6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	$\bar{1}$·7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	$\bar{1}$·7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	$\bar{1}$·7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	$\bar{1}$·7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	$\bar{1}$·7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	$\bar{1}$·7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	$\bar{1}$·7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	$\bar{1}$·7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	$\bar{1}$·7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40	$\bar{1}$·8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	$\bar{1}$·8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	$\bar{1}$·8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	$\bar{1}$·8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	$\bar{1}$·8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	$\bar{1}$·8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	$\bar{1}$·8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	$\bar{1}$·8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	$\bar{1}$·8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	$\bar{1}$·8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50	$\bar{1}$·8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	$\bar{1}$·8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	$\bar{1}$·8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	$\bar{1}$·9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	$\bar{1}$·9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	$\bar{1}$·9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	$\bar{1}$·9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	$\bar{1}$·9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	$\bar{1}$·9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	$\bar{1}$·9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60	$\bar{1}$·9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	$\bar{1}$·9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	$\bar{1}$·9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	$\bar{1}$·9499	9503	9507	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	$\bar{1}$·9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	$\bar{1}$·9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	$\bar{1}$·9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	$\bar{1}$·9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	$\bar{1}$·9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	$\bar{1}$·9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70	$\bar{1}$·9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	$\bar{1}$·9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	$\bar{1}$·9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	$\bar{1}$·9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	2	2
74	$\bar{1}$·9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	$\bar{1}$·9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	$\bar{1}$·9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	$\bar{1}$·9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	$\bar{1}$·9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	$\bar{1}$·9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80	$\bar{1}$·9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	$\bar{1}$·9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	$\bar{1}$·9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83	$\bar{1}$·9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	0	1	1
84	$\bar{1}$·9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85	$\bar{1}$·9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86	$\bar{1}$·9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87	$\bar{1}$·9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88	$\bar{1}$·9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89	$\bar{1}$·9999	9999	0·0000	0000	0000	0000	0000	0000	0000	0000					
90	0·0000														

(*Subtract Mean Differences*)

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	0·0000	0000	0000	0000	0000	0000	0000	0000	0000	$\bar{1}$·9999	0	0	0	0	0
1	$\bar{1}$·9999	9999	9999	9999	9999	9999	9998	9998	9998	9998	0	0	0	0	0
2	$\bar{1}$·9997	9997	9997	9996	9996	9996	9996	9995	9995	9994	0	0	0	0	0
3	$\bar{1}$·9994	9994	9993	9993	9992	9992	9991	9991	9990	9990	0	0	0	0	0
4	$\bar{1}$·9989	9989	9988	9988	9987	9987	9986	9985	9985	9984	0	0	0	0	0
5	$\bar{1}$·9983	9983	9982	9981	9981	9980	9979	9978	9978	9977	0	0	0	0	1
6	$\bar{1}$·9976	9975	9975	9974	9973	9972	9971	9970	9969	9968	0	0	0	1	1
7	$\bar{1}$·9968	9967	9966	9965	9964	9963	9962	9961	9960	9959	0	0	1	1	1
8	$\bar{1}$·9958	9956	9955	9954	9953	9952	9951	9950	9949	9947	0	0	1	1	1
9	$\bar{1}$·9946	9945	9944	9943	9941	9940	9939	9937	9936	9935	0	0	1	1	1
10	$\bar{1}$·9934	9932	9931	9929	9928	9927	9925	9924	9922	9921	0	0	1	1	1
11	$\bar{1}$·9919	9918	9916	9915	9913	9912	9910	9909	9907	9906	0	1	1	1	1
12	$\bar{1}$·9904	9902	9901	9899	9897	9896	9894	9892	9891	9889	0	1	1	1	1
13	$\bar{1}$·9887	9885	9884	9882	9880	9878	9876	9875	9873	9871	0	1	1	1	2
14	$\bar{1}$·9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15	$\bar{1}$·9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16	$\bar{1}$·9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	2	2
17	$\bar{1}$·9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18	$\bar{1}$·9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19	$\bar{1}$·9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20	$\bar{1}$·9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21	$\bar{1}$·9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22	$\bar{1}$·9672	9669	9666	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23	$\bar{1}$·9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24	$\bar{1}$·9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25	$\bar{1}$·9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26	$\bar{1}$·9537	9533	9529	9525	9522	9518	9514	9510	9507	9503	1	1	2	3	3
27	$\bar{1}$·9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28	$\bar{1}$·9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29	$\bar{1}$·9418	9414	9410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30	$\bar{1}$·9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31	$\bar{1}$·9331	9326	9322	9317	9312	9308	9303	9298	9294	9289	1	2	2	3	4
32	$\bar{1}$·9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33	$\bar{1}$·9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34	$\bar{1}$·9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35	$\bar{1}$·9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36	$\bar{1}$·9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37	$\bar{1}$·9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38	$\bar{1}$·8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39	$\bar{1}$·8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40	$\bar{1}$·8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41	$\bar{1}$·8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	5	6
42	$\bar{1}$·8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43	$\bar{1}$·8641	8634	8627	8620	8613	8606	8598	8591	8584	8577	1	2	3	5	6
44	$\bar{1}$·8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6

LOGARITHMS OF COSINES
(Subtract Mean Differences)

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
45	$\bar{1}$·8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5	6
46	$\bar{1}$·8418	8410	8402	8394	8386	8378	8370	8362	8354	8346	1	3	4	5	7
47	$\bar{1}$·8338	8330	8322	8313	8305	8297	8289	8280	8272	8264	1	3	4	6	7
48	$\bar{1}$·8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6	7
49	$\bar{1}$·8169	8161	8152	8143	8134	8125	8117	8108	8099	8090	1	3	4	6	7
50	$\bar{1}$·8081	8072	8063	8053	8044	8035	8026	8017	8007	7998	2	3	5	6	8
51	$\bar{1}$·7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6	8
52	$\bar{1}$·7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7	8
53	$\bar{1}$·7795	7785	7774	7764	7754	7744	7734	7723	7713	7703	2	3	5	7	9
54	$\bar{1}$·7692	7682	7671	7661	7650	7640	7629	7618	7607	7597	2	4	5	7	9
55	$\bar{1}$·7586	7575	7564	7553	7542	7531	7520	7509	7498	7487	2	4	6	7	9
56	$\bar{1}$·7476	7464	7453	7442	7430	7419	7407	7396	7384	7373	2	4	6	8	10
57	$\bar{1}$·7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8	10
58	$\bar{1}$·7242	7230	7218	7205	7193	7181	7168	7156	7144	7131	2	4	6	8	10
59	$\bar{1}$·7118	7106	7093	7080	7068	7055	7042	7029	7016	7003	2	4	6	9	11
60	$\bar{1}$·6990	6977	6963	6950	6937	6923	6910	6896	6883	6869	2	4	7	9	11
61	$\bar{1}$·6856	6842	6828	6814	6801	6787	6773	6759	6744	6730	2	5	7	9	12
62	$\bar{1}$·6716	6702	6687	6673	6659	6644	6629	6615	6600	6585	2	5	7	10	12
63	$\bar{1}$·6570	6556	6541	6526	6510	6495	6480	6465	6449	6434	3	5	8	10	13
64	$\bar{1}$·6418	6403	6387	6371	6356	6340	6324	6308	6292	6276	3	5	8	11	13
65	$\bar{1}$·6259	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11	14
66	$\bar{1}$·6093	6076	6059	6042	6024	6007	5990	5972	5954	5937	3	6	9	12	15
67	$\bar{1}$·5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12	15
68	$\bar{1}$·5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13	16
69	$\bar{1}$·5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14	17
70	$\bar{1}$·5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14	18
71	$\bar{1}$·5126	5104	5082	5060	5037	5015	4992	4969	4946	4923	4	8	11	15	19
72	$\bar{1}$·4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16	20
73	$\bar{1}$·4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	4	9	13	17	21
74	$\bar{1}$·4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	5	9	14	18	23
75	$\bar{1}$·4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	5	10	15	20	24
76	$\bar{1}$·3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	5	11	16	21	26
77	$\bar{1}$·3521	3488	3455	3421	3387	3353	3319	3284	3250	3214	6	11	17	23	28
78	$\bar{1}$·3179	3143	3107	3070	3034	2997	2959	2921	2883	2845	6	12	19	25	31
79	$\bar{1}$·2806	2767	2727	2687	2647	2606	2565	2524	2482	2439	7	14	20	27	34
80	$\bar{1}$·2397	2353	2310	2266	2221	2176	2131	2085	2038	1991	8	15	23	30	38
81	$\bar{1}$·1943	1895	1847	1797	1747	1697	1646	1594	1542	1489	8	17	25	34	42
82	$\bar{1}$·1436	1381	1326	1271	1214	1157	1099	1040	0981	0920	10	19	29	38	48
83	$\bar{1}$·0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	11	22	33	44	55
84	$\bar{1}$·0192	0120	0046	$\bar{2}$·9970	$\bar{2}$·9894	$\bar{2}$·9816	$\bar{2}$·9736	$\bar{2}$·9655	$\bar{2}$·9573	$\bar{2}$·9489	13	26	39	52	65
85	$\bar{2}$·9403	9315	9226	9135	9042	8946	8849	8749	8647	8543	16	32	48	64	80
86	$\bar{2}$·8436	8326	8213	8098	7979	7857	7731	7602	7468	7330					
87	$\bar{2}$·7188	7041	6889	6731	6567	6397	6220	6035	5842	5640					
88	$\bar{2}$·5428	5206	4971	4723	4459	4179	3880	3558	3210	2832					
89	$\bar{2}$·2419	1961	1450	0870	0200	$\bar{3}$·9408	$\bar{3}$·8439	$\bar{3}$·7190	$\bar{3}$·5429	$\bar{3}$·2419					
90															

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences 1	2	3	4	5
0	$-\infty$	$\bar{3}$·2419	$\bar{3}$·5429	$\bar{3}$·7190	$\bar{3}$·8439	$\bar{3}$·9409	$\bar{2}$·0200	$\bar{2}$·0870	$\bar{2}$·1450	$\bar{2}$·196[illegible]					
1	$\bar{2}$·2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	$\bar{2}$·5431	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3	$\bar{2}$·7194	7337	7475	7609	7739	7865	7988	8107	8223	833[illegible]					
4	$\bar{2}$·8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5	$\bar{2}$·9420	9506	9591	9674	9756	9836	9915	9992	$\bar{1}$·0068	$\bar{1}$·0143	13	26	40	53	66
6	$\bar{1}$·0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	$\bar{1}$·0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	$\bar{1}$·1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	$\bar{1}$·1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10	$\bar{1}$·2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	[illegible]	28	35
11	$\bar{1}$·2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	[illegible]	32
12	$\bar{1}$·3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	[illegible]	30
13	$\bar{1}$·3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	$\bar{1}$·3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	$\bar{1}$·4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	$\bar{1}$·4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	$\bar{1}$·4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	$\bar{1}$·5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	$\bar{1}$·5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20	$\bar{1}$·5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	$\bar{1}$·5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	[illegible]	19
22	$\bar{1}$·6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	[illegible]	18
23	$\bar{1}$·6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	[illegible]
24	$\bar{1}$·6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	$\bar{1}$·6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	[illegible]
26	$\bar{1}$·6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	$\bar{1}$·7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	$\bar{1}$·7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	$\bar{1}$·7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	[illegible]	15
30	$\bar{1}$·7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	$\bar{1}$·7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	$\bar{1}$·7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	$\bar{1}$·8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	$\bar{1}$·8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	$\bar{1}$·8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	$\bar{1}$·8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	$\bar{1}$·8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	$\bar{1}$·8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	$\bar{1}$·9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40	$\bar{1}$·9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	$\bar{1}$·9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	$\bar{1}$·9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	$\bar{1}$·9697	9712	9727	9742	9757	9773	9788	9803	9818	9833	3	5	8	10	13
44	$\bar{1}$·9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

Degrees	0′ 0°·0	6′ 0°·1	12′ 0°·2	18′ 0°·3	24′ 0°·4	30′ 0°·5	36′ 0°·6	42′ 0°·7	48′ 0°·8	54′ 0°·9	Mean Differences. 1	2	3	4	5
45	·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136	3	5	8	10	13
46	·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288	3	5	8	10	13
47	·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440	3	5	8	10	13
48	·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593	3	5	8	10	13
49	·0608	0624	0639	0654	0670	0685	0700	0716	0731	0746	3	5	8	10	13
50	·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901	3	5	8	10	13
51	·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056	3	5	8	10	13
52	·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213	3	5	8	10	13
53	·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371	3	5	8	11	13
54	·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532	3	5	8	11	13
55	·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694	3	5	8	11	14
56	·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858	3	5	8	11	14
57	·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025	3	6	8	11	14
58	·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195	3	6	9	11	14
59	·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368	3	6	9	12	14
60	·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545	3	6	9	12	15
61	·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725	3	6	9	12	15
62	·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910	3	6	9	12	15
63	·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099	3	6	9	13	16
64	·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294	3	6	10	13	16
65	·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494	3	7	10	13	17
66	·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700	3	7	10	14	17
67	·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914	4	7	11	14	18
68	·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136	4	7	11	15	19
69	·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366	4	8	12	15	19
70	·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606	4	8	12	16	20
71	·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857	4	8	13	17	21
72	·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120	4	9	13	18	22
73	·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397	5	9	14	19	23
74	·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689	5	10	15	20	25
75	·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000	5	10	16	21	26
76	·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332	6	11	17	22	28
77	·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688	6	12	18	24	30
78	·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073	6	13	19	26	32
79	·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493	7	14	21	28	35
80	·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954	8	16	23	31	39
81	·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467	9	17	26	35	43
82	·8522	8577	8633	8690	8748	8806	8865	8924	8985	9046	10	20	29	39	49
83	·9109	9172	9236	9301	9367	9433	9501	9570	9640	9711	11	22	34	45	56
84	·9784	9857	9932	1·0008	1·0085	1·0164	1·0244	1·0326	1·0409	1·0494	13	26	40	53	66
85	1·0580	0669	0759	0850	0944	1040	1138	1238	1341	1446	16	32	48	64	81
86	1·1554	1664	1777	1893	2012	2135	2261	2391	2525	2663					
87	1·2806	2954	3106	3264	3429	3599	3777	3962	4155	4357					
88	1·4569	4792	5027	5275	5539	5819	6119	6441	6789	7167					
89	1·7581	8038	8550	9130	9800	2·0591	2·1561	2·2810	2·4571	2·7581					